Ex 1: $\dfrac{1}{x+2} + \dfrac{1}{x-3} = \dfrac{x-3 + x+2}{(x+2)(x-3)} = \dfrac{2x-1}{x^2-x-6}$

So

$$\int \frac{2x-1}{x^2-x-6}\,dx = \int\left[\frac{1}{x+2} + \frac{1}{x-3}\right]dx = \ln|x+2| + \ln|x-3| + c$$

$$= \ln\left|(x+2)(x-3)\right| + c.$$

Ex 2: $\displaystyle\int \frac{x^2+2x}{x-3}\,dx$ use poly. long div. to write w/ deg of num. less than the deg. of den.

$$\begin{array}{r l}
 & x + 5 \\
x - 3 \,\big)\!\!\! & \overline{x^2 + 2x + 0} \\
 & -(x^2 - 3x) \\
 & \overline{\quad 5x + 0} \\
 & -(5x - 15) \\
 & \overline{\qquad 15}
\end{array}$$

$$= \int\left[x + 5 + \frac{15}{x-3}\right]dx$$

$$= \frac{x^2}{2} + 5x + 15\ln|x-3| + c.$$

Work on your own: $\displaystyle\int \frac{1-x^2}{x+7}\,dx = \int\left[-x + 7 - \frac{48}{x+7}\right]dx$

$$\begin{array}{r l}
 & -x + 7 \\
x + 7 \,\big)\!\!\! & \overline{-x^2 + 0x + 1} \\
 & -(-x^2 - 7x) \\
 & \overline{\quad 7x + 1} \\
 & -(7x + 49) \\
 & \overline{\qquad -48}
\end{array}$$

$$= -\frac{x^2}{2} + 7x - 48\ln|x+7| + c$$

Ex 3: $\int \frac{x-1}{\underbrace{x^2+9x+20}} \, dx$

$$\frac{x-1}{x^2+9x+20} = \frac{x-1}{\underbrace{(x+5)(x+4)}_{\text{product of unique linear factors.}}} = \frac{A}{(x+5)} + \frac{B}{x+4}$$

$$\Rightarrow \quad x-1 = A(x+4) + B(x+5)$$

$$\Rightarrow \quad x-1 = Ax + 4A + Bx + 5B$$

$$\Rightarrow \quad \begin{cases} A + B = 1 \\ 4A + 5B = -1 \end{cases} \Rightarrow B = -5 \text{ and } A = 6.$$

$$= \int \left[\frac{6}{x+5} - \frac{5}{x+4}\right] dx$$

$$= 6 \ln|x+5| - 5\ln|x+4| + C.$$

Ex 4: $\displaystyle\int \frac{1-2x^2}{x^3+4x^2+4x}\,dx = \int \frac{1-2x^2}{x(x^2+4x+4)}\,dx = \int \underbrace{\frac{1-2x^2}{x\,(x+2)^2}}\,dx$

product of linear terms w/ some repetition.

$$\frac{1-2x^2}{x(x+2)^2} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

$$\Rightarrow\ 1-2x^2 = A(x+2)^2 + Bx(x+2) + Cx$$

$$\Rightarrow\ 1-2x^2 = Ax^2 + 4Ax + 4A + Bx^2 + 2Bx + Cx$$

$$\Rightarrow \begin{cases} A+B = -2 \\ 4A+2B+C = 0 \\ 4A = 1 \end{cases} \qquad B = -\frac{9}{4} \quad C = 7/2 \quad A = \frac{1}{4}$$

$$= \int\left[\frac{1}{4}\cdot\frac{1}{x} - \frac{9}{4}\cdot\frac{1}{x+2} + \frac{7}{2}\cdot\frac{1}{(x+2)^2}\right]dx$$

$$= \frac{1}{4}\ln|x| - \frac{9}{4}\ln|x+2| - \frac{7}{2}\cdot\frac{1}{(x+2)} + C_1.$$

work on your own: $\displaystyle\int \frac{5x}{x^2-6x+9}\,dx = \int \frac{5x}{(x-3)^2}\,dx$

$$\frac{5x}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2}$$

$\Rightarrow\ 5x = A(x-3) + B$

$\Rightarrow\ 5 = A$

$0 = -3A + B \ \leftarrow\ B = 15$

$$= \int\left[\frac{5}{x-3} + \frac{15}{(x-3)^2}\right]dx$$

$$= 5\ln|x-3| - 15\cdot\frac{1}{x-3} + C_1.$$

Ex 5: $\displaystyle \int \frac{x-1}{x^3+3x}\,dx = \int \underbrace{\frac{x-1}{x(x^2+3)}}\,dx$

containing an irreducible quadratic factor.

$$\frac{x-1}{x(x^2+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+3}$$

$$\Rightarrow x-1 = A(x^2+3) + (Bx+C)\,x$$

$$\Rightarrow \begin{cases} B + A = 0 \quad \Rightarrow \quad B = 1/3 \\ C = 1 \\ 3A = -1 \quad \Rightarrow \quad A = -1/3 \end{cases}$$

$$= \int \left[-\frac{1}{3}\cdot\frac{1}{x} + \frac{\frac{1}{3}x + 1}{x^2+3} \right] dx$$

Let $u = x^2 + 3$
$du = 2x\,dx$

$$= -\frac{1}{3}\ln|x| + \underbrace{\frac{1}{3}\int \frac{x}{x^2+3}\,dx}_{\frac{1}{6}\ln|x^2+3|} + \int \frac{1}{x^2+3}\,dx$$

$$\frac{1}{6}\ln|x^2+3| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C.$$

Ex 6: $\displaystyle\int \frac{x^3 - 5x + 1}{x^4 + 4x^2 + 4}\,dx$

$$= \int \underbrace{\frac{x^3 - 5x + 1}{(x^2+2)^2}}_{\text{product of irreducible quadratic terms not unique.}}\,dx$$

$$\frac{x^3 - 5x + 1}{(x^2+2)^2} = \frac{Ax + B}{x^2 + 2} + \frac{Cx + D}{(x^2+2)^2}$$

$$\Rightarrow \quad x^3 - 5x + 1 = (Ax + B)(x^2 + 2) + Cx + D$$

$$\Rightarrow \quad x^3 - 5x + 1 = Ax^3 + 2Ax + Bx^2 + 2B + Cx + D.$$

$$\begin{array}{c} \\ x^3 \\ x^2 \\ x \\ 1 \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & D & = & \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & -5 \\ 0 & 2 & 0 & 1 & 1 \end{array}\right] \end{array} \qquad \begin{array}{l} A = 1 \\ B = 0 \\ C = -7 \\ D = 1 \end{array}$$

$$\int \frac{x^3 - 5x + 1}{x^4 + 4x^2 + 4}\,dx = \int \left[\underbrace{\frac{1}{x^2 + 2}}_{\arctan\left(\frac{x}{\sqrt{2}}\right)} + \frac{-7x + 1}{(x^2+2)^2}\right] dx$$

$u = x^2 + 2$

$\frac{du}{2} = 2x\,dx$

use trig substitution

$$\underbrace{\int \frac{-7x}{(x^2+2)^2}\,dx}_{\frac{7}{2}\cdot\frac{1}{x^2+2}} + \underbrace{\int \frac{1}{(x^2+2)^2}\,dx}_{\frac{\sqrt{2}}{8}\arctan\left(\frac{x}{\sqrt{2}}\right) + \frac{x}{4(x^2+2)}}$$

(see attached)

(attached)

scratch work for example 6

To find $\int \frac{1}{(x^2+2)^2}\,dx$

Let $x = \sqrt{2}\tan(\theta)$

$dx = \sqrt{2}\sec^2(\theta)\,d\theta.$

$$= \int \frac{1}{(2\tan^2(\theta)+2)^2}\cdot\sqrt{2}\sec^2(\theta)\,d\theta.$$

$$= \frac{\sqrt{2}}{4}\int \frac{\sec^2(\theta)}{(\tan^2(\theta)+1)^2}\,d\theta$$

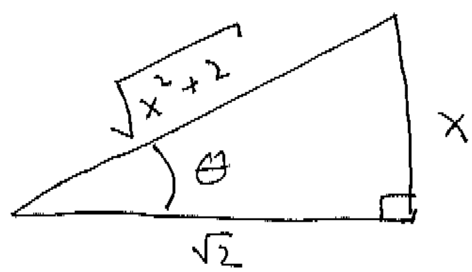
$$= \frac{\sqrt{2}}{4}\int \frac{\sec^2(\theta)}{\sec^4(\theta)}\,d\theta.$$

$$= \frac{\sqrt{2}}{4}\int \cos^2(\theta)\,d\theta$$

$$= \frac{\sqrt{2}}{4}\int \frac{1+\cos(2\theta)}{2}\,d\theta.$$

$$= \frac{\sqrt{2}}{8}\left[\theta + \frac{\sin(2\theta)}{2}\right]$$

$$= \frac{\sqrt{2}}{8}\left[\theta + \sin(\theta)\cos(\theta)\right]$$



$$= \frac{\sqrt{2}}{8}\left[\arctan\left(\frac{x}{\sqrt{2}}\right) + \frac{x}{\sqrt{x^2+2}}\cdot\frac{\sqrt{2}}{\sqrt{x^2+2}}\right]$$

$$= \frac{\sqrt{2}}{8}\arctan\left(\frac{x}{\sqrt{2}}\right) + \frac{x}{4(x^2+2)}.$$

8.4
7/7

Group Work 1, Section 8.4
Partial Fractions (Version 3)

from Teaching supplements.

1. Compute the following integrals:

(a) $\displaystyle\int \frac{dx}{x+1}$

(b) $\displaystyle\int \frac{dx}{x+2}$

(c) $\displaystyle\int \frac{dx}{x^2+4}$

2. Factor $x^4 + 3x^3 + 6x^2 + 12x + 8$. (Hint: see above)

3. Compute $\displaystyle\int \frac{20x^2\,dx}{x^4 + 3x^3 + 6x^2 + 12x + 8}$.

4. Compute $\displaystyle\int \frac{x^4 + 3x^3 + 26x^2 + 12x + 8}{x^4 + 3x^3 + 6x^2 + 12x + 8}\,dx$. (Hint: what is the degree of the numerator & denominator?).