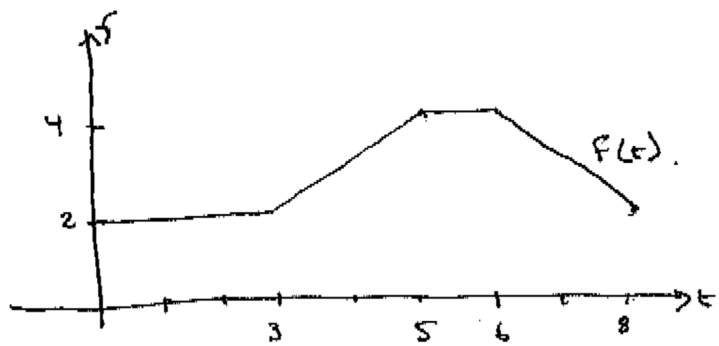
5.3 : The Fundamental Theorem of Calculus

Recall the defn of the definite integral

$$\underbrace{\int_a^b f(x)\,dx}_{\text{number}} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

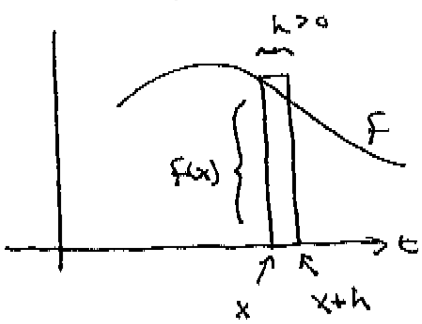
To "functionize" this, $g(x) = \int_a^x f(t)\,dt$.

Ex1: Sketch $g(x)$ for the given graph of $f(t)$.



$$g(x) = \int_0^x f(t)\,dt.$$

Let's explore the relationship between f and g.



$$g(x+h) - g(x) \approx h \cdot f(x)$$

$$\Rightarrow \frac{g(x+h) - g(x)}{h} \approx f(x)$$

intuitively

$$\Rightarrow \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = g'(x) = f(x)$$

The Fundamental Theorem of Calculus, part 1.

If f is cont on $[a,b]$, then the fct g defined by $g(x) = \int_a^x f(t)\,dt$, $a \le x \le b$ is cont on $[a,b]$ and diff on (a,b), and $g'(x) = f(x)$.

5.3
2/15

In our proof of FToC 1, we need to recall the following.

I) EVT: If f is cont on $[a,b]$, then f attains an abs max $M = f(u)$ and an abs min $m = f(v)$ for some $u, v \in [a,b]$

II) Comp. Prop: If $m \leq f(x) \leq M$ on $[a,b]$, then
$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

III) The Squeeze Thm: If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.

IV) Differentiability: If f is differentiable at a, then f is continuous at a.

□ proof of FToC 1.

Assume f is cont on $[a,b]$ and $g(x) = \int_a^x f(t)dt$.

If $x, x+h \in (a,b)$, then
$$g(x+h) - g(x) = \int_a^{x+h} f(t)dt - \int_a^x f(t)dt$$
$$= \left(\int_a^x f(t)dt + \int_x^{x+h} f(t)dt\right) - \int_a^x f(t)dt.$$
$$= \int_x^{x+h} f(t)dt.$$

and so, for $h \neq 0$
$$\frac{g(x+h) - g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)dt$$

WLOG, assume $h>0$. Since f is cont on $[x, x+h]$, the EVT guarantees that $\exists\, u, v \in [x, x+h]$ s.t. $f(u) = M$ and $f(v) = m$ where m, M are the abs min & max of f on $[x, x+h]$

$$\Rightarrow \quad m \leq f(x) \leq M \quad \text{on } [x, x+h]$$

$$\Rightarrow \quad mh \leq \int_x^{x+h} f(x)\,dx \leq Mh \quad \text{by the integral property.}$$

$\Rightarrow$ since $h>0$, we have

$$m \leq \frac{1}{h}\int_x^{x+h} f(x)\,dx \leq M$$

$$\text{OR} \quad f(v) \leq \underbrace{\frac{1}{h}\int_x^{x+h} f(x)\,dx}_{} \leq f(u)$$

$$\Rightarrow \quad f(v) \leq \frac{g(x+h)-g(x)}{h} \leq f(u).$$

It can also be shown that this inequality holds when $h<0$.

Now $\displaystyle\lim_{h\to 0} f(v) = \lim_{v\to x} f(v) = f(x)$ and $\displaystyle\lim_{h\to 0} f(u) = f(x)$

so $\displaystyle\lim_{h\to 0} f(v) \leq \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \leq \lim_{h\to 0} f(u)$

and by the squeeze thm $\displaystyle\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = g'(x) = f(x)$.

Allowing for 1-sided limits at a & b, $g'(x)$ exists on $[a,b]$, so g is continuous on $[a,b]$. $\blacksquare$

We can write FToC1 concisely as

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Ex 2: Find the derivative of $g(x) = \int_0^x \cos(\sin(\ln(t)))\,dt$.

An example is the ~~Erf~~ error fct from statistics

$$Erf(x) = \int_0^x \frac{2}{\sqrt{\pi}} e^{-t^2}\,dt.$$

If $u(x) = x^2$

Ex 3: $$\frac{d}{dx}\int_1^{x^2} \frac{\cos(t)}{t}\,dt = \frac{d}{du}\int_1^{u(x)} \frac{\cos(t)}{t}\,dt\,\frac{du}{dx}$$

If $u(x) = \sin x$

Ex 4: $$\frac{d}{dx}\int_0^{\sin x} (t + \cos(t))\,dt = \frac{d}{du}\int_0^{u} (t + \cos(t))\,dt\,\frac{du}{dx}$$

The Fundamental Theorem of Calculus, part 2.

If f is cont on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any anti-derivative of f, that is a fct such that $F' = f$.

Ex 5: Evaluate $\int_0^{\pi} \sin x\,dx$.

5.3
5/5

Proof of FToC 2:

Let $g(x) = \int_a^x f(t)\,dt$. From FToC 1, we know that $g' = f$, that is, g is an antiderivative of f. If F is any other antiderivative of f on $[a,b]$, we know that F and g differ by a constant c.

$$* \qquad F(x) = g(x) + c. \quad \text{for } a < x < b.$$

But both F & g are cont on $[a,b]$ and so taking one sided limits of $*$, we see that it holds for $[a,b]$. Letting $x=a$ & $x=b$ in $*$:

$$\begin{aligned} F(b) - F(a) &= (g(b)+c) - (g(a)+c) \\ &= g(b) - g(a) \\ &= g(b), \text{ since } g(a) = \int_a^a f(t)\,dt = 0 \\ &= \int_a^b f(t)\,dt \quad \blacksquare \end{aligned}$$

Ex6: $\int_0^1 x^3\,dx$

Ex7: $\int_{-2}^{2} \frac{1}{x^3}\,dx$ (what's wrong?)

Ex 8: $\int_8^{24} \frac{dx}{x}$