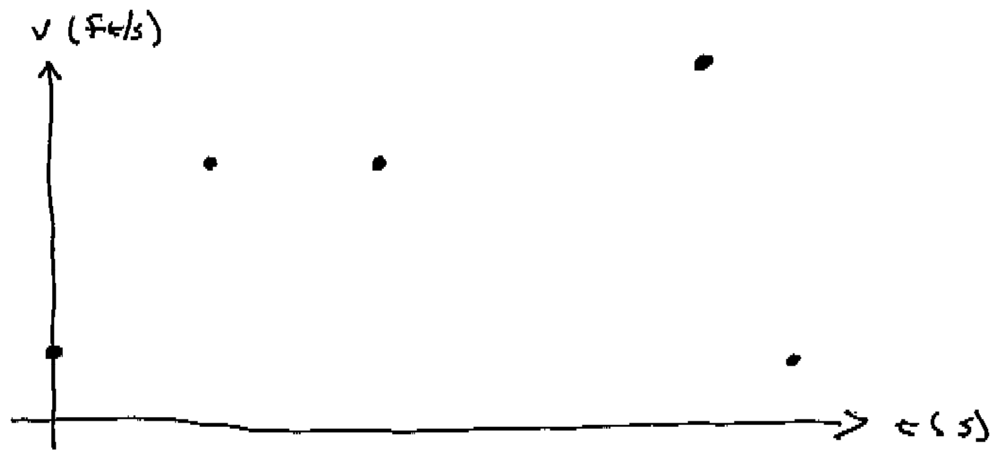
5.1
1/5

Ex 1: An example to motivate areas.

During the last snow storm, I found myself driving S on Pac. Hwy in my 1989 Ford Ranger w/o chains & w/o weight in the back. I was white knuckling it along & glancing down @ my custom speedometer every now & then... just to make sure I was still alive.

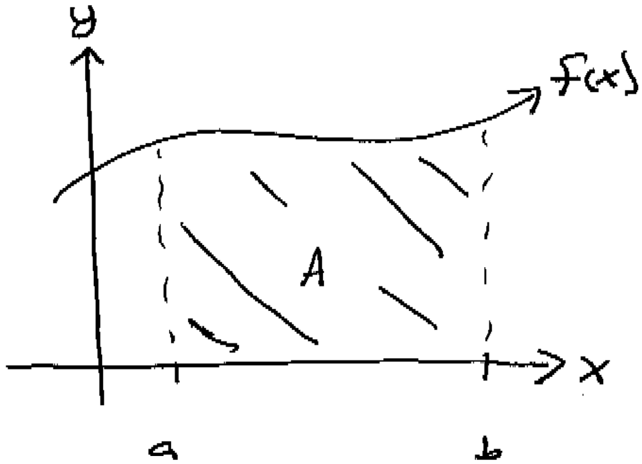
t (sec)	0	60	120	240	270	300	360	420
v (ft/s)	10	35	35	50	10	-20	20	0

About how far did I travel?



Use left-endpoints, right, & their average (the trapezoid rule).

So, generalizing from the example, our goal is to find the area "under" an arbitrary curve $f(x)$.



Find the area A of the region bounded by f $[f \geq 0]$, $x=a$, $x=b$, and the x-axis.

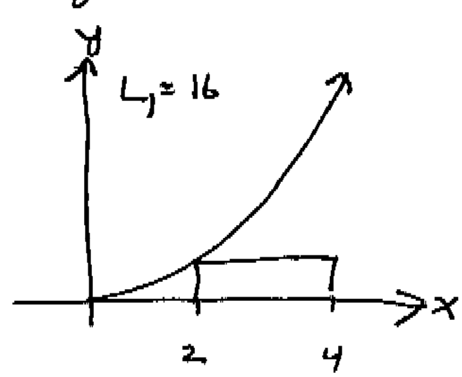
NOTE: If our axes were measured w/ time & velocity, the area would represent a distance traveled.

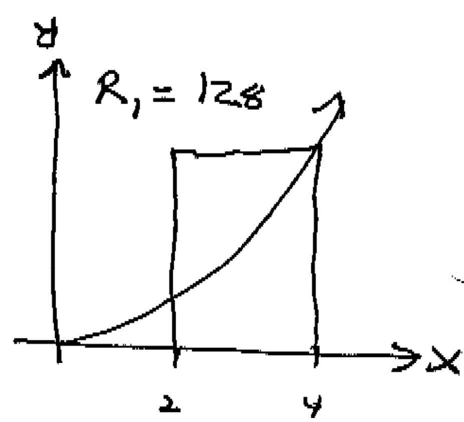
Our basic method: we will approximate the area A w/ rectangles.

Ex2: Consider the region under $y=x^3$ on $2 \leq x \leq 4$. Approx the area A of the region using rectangles.

(A) 1 rect.

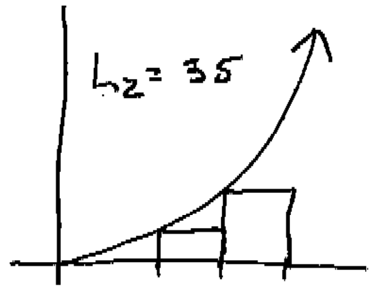
$16 \leq A \leq 128$

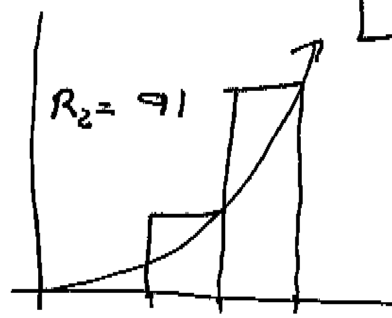




(B) 2 rect.

$35 \le A \le 91$



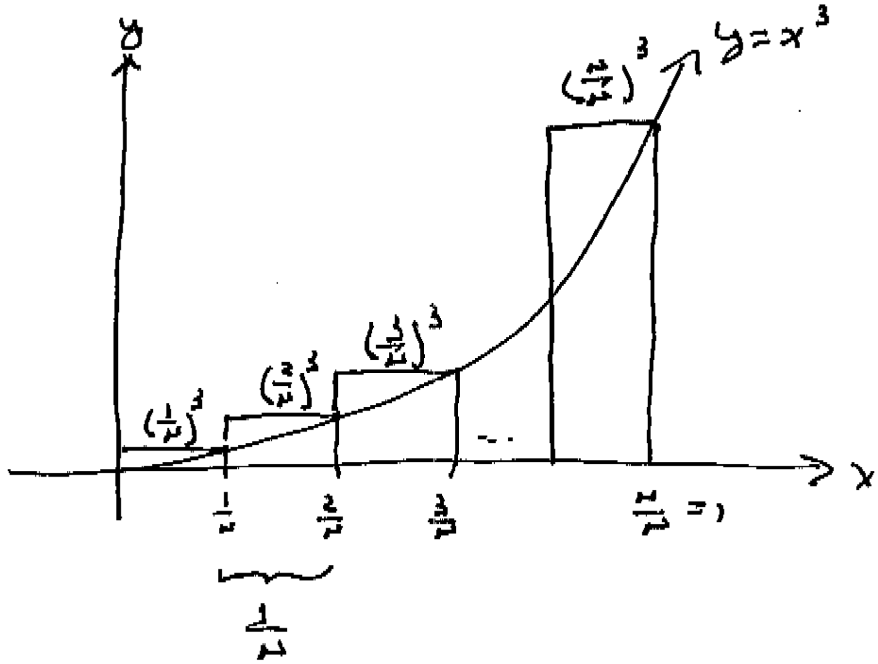


mathematica ...

key point: $\lim_{n \to \infty} L_n = A$ & $\lim_{n\to\infty} R_n = A$.

To show how this works w/o becoming overly bogged down in details.

<u>Ex 2:</u> Find the area under $y = x^3$ on $0 \le x \le 1$ using rectangles w/ right endpts & n rectangles of equal width. Then, find A.



$$R_n = \frac{1}{n}\left(\frac{1}{n}\right)^3 + \frac{1}{n}\left(\frac{2}{n}\right)^3 + \frac{1}{n}\left(\frac{3}{n}\right)^3 + \dots + \frac{1}{n}\left(\frac{n}{n}\right)^3$$

$$= \frac{1}{n^4}\underbrace{\left(1^3 + 2^3 + \dots + n^3\right)}$$

Don't we all know this?

$$= \frac{1}{n^4} \cdot \frac{1}{4}n^2(n+1)^2$$

$$= \frac{1}{n^4} \cdot \frac{1}{4}\left(n^4 + 2n^3 + n^2\right)$$

$$= \frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2}$$

AND $$A = \lim_{n\to\infty} R_n$$

$$= \lim_{n\to\infty}\left(\frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2}\right)$$

$$= \frac{1}{4}$$

Conclusion! The exact area under $y = x^3$ on $0 \le x \le 1$ is $\frac{1}{4}$.

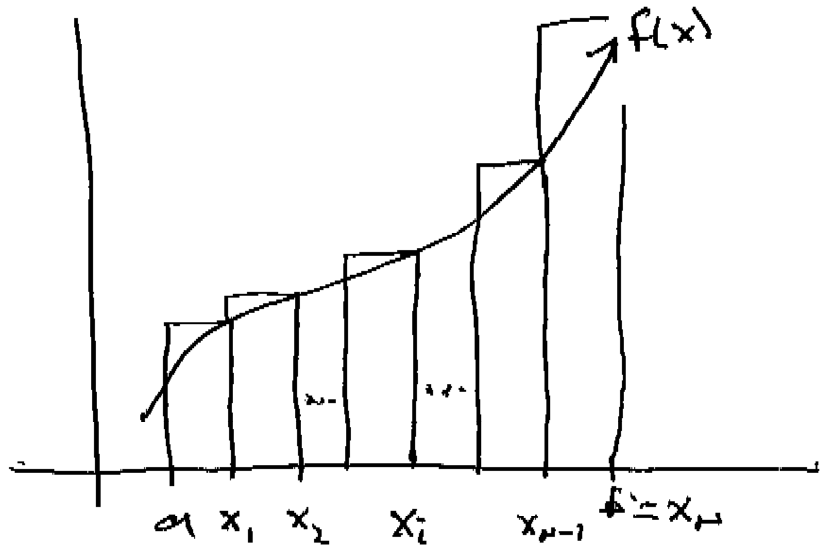
NOTATION: (A) $1^3 + 2^3 + \dots + n^3 = \sum_{i=1}^{n} i^3$

(B) $\sum_{i=1}^{10} \frac{i-1}{i}$

(C) $x_0 + x_1 + \dots + x_{n-1} = \sum_{i=0}^{n-1} x_i = \sum_{i=1}^{n} x_{i-1}$

Generallizing the procedure of ex2 and using the sigma notation:

To find the area under $y=f(x)$ from $x=a$ to $x=b$ using n rectangles of equal width...



Let $\Delta x = \frac{b-a}{n}$

and $a = x_0$

$b = x_n$

so $x_1 = a + \Delta x$

$x_2 = a + 2\Delta x$

$x_i = a + i\Delta x$

$\vdots$

right endpoints.

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_i)\Delta x + \dots + f(x_n)\Delta x$$

$$= \sum_{i=1}^{n} f(x_i)\Delta x$$

and $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

left endpoints

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

In general

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \quad \text{where } x_{i-1} \le x_i^* \le x_i$$

for $i \in \{1, 2, \dots, n\}$