Test 3
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Math 125 – Spring 2005

No work = no credit

Name: Key

I know not what I appear to the world, but to myself I seem to have been only like a boy playing on the sea-shore, and diverting myself in now and then finding a smoother pebble or a prettier shell, whilest the great ocean of truth lay all undiscovered before me.

Isaac Newton (1643 - 1727)
English mathematician

Warm-ups (1 pt each) $-\pi^0 =$ -1 $\frac{\pi}{0} =$ undefined $\cos(\pi) =$ -1

1.) (1 pt) According to the quote (see above), what portion of truth did Newton discover? Answer using complete sentences.

Newton discovered pebbles on the ocean of truth.

Solution: ______________________

2.) (10 pts) Use the Comparison Theorem to determine whether the integral $\int_0^8 \frac{|\cos(x)|}{\sqrt[3]{x}}\,dx$ converges or diverges.

$$\frac{|\cos x|}{\sqrt[3]{x}} \leq \frac{1}{\sqrt[3]{x}} \quad \text{and} \quad \int_0^8 \frac{dx}{x^{1/3}} = \lim_{t\to 0^+} \int_t^8 x^{-1/3}\,dx$$

$$= \lim_{t\to 0^+} \frac{3}{2} x^{2/3} \Big|_t^8$$

$$= \lim_{t\to 0^+} \left(\frac{3}{2}(4) - \frac{3}{2}t^{2/3}\right)$$

$$= 6$$

3 pts for improper.

Solution: So $\int_0^8 \frac{|\cos x|}{\sqrt[3]{x}}\,dx$ converges by comparison.

1 pt conclusion.

3.) (10 pts) Complete one of the follow two problems (That is, either (a.) or (b.))

a.) Use Part 1 of the Fundamental Theorem of Calculus to find $\frac{d}{dx}\int_1^{\sin(x)} \arcsin(u)\,du$.

6pts

$\underbrace{\arcsin(\sin x)}_{x} \cdot \cos x$

Solution: $x \cos x$

b.) Use the definition of the definite integral to evaluate $\int_0^5 (3x+2)\,dx$ given that the sum $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. You may check your answer using FToC2.

$\Delta x = \frac{5}{n}$

$x_i = \frac{5i}{n}$

2pts

$$\int_0^5 (3x+2)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(3\left(\frac{5i}{n}\right) + 2\right)\frac{5}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{75i}{n^2} + \frac{10}{n}\right)$$

$$= \lim_{n\to\infty} \left(\frac{75}{n^2} \cdot \frac{n(n+1)}{2} + \frac{10}{n} \cdot n\right)$$

$$= 75/2 + 10$$

check

$$\left.\frac{3x^2}{2} + 2x\right|_0^5$$

$$= \frac{75}{2} + 10$$

Solution: $95/2$

4.) (10 pts) For what values of r does $y = e^{rt}$ satisfy the differential equation $y''+3y'-4y=0$?

$y' = re^{rt}$ & $y'' = r^2e^{rt}$

$\Rightarrow r^2e^{rt} + 3re^{rt} - 4e^{rt} = 0$

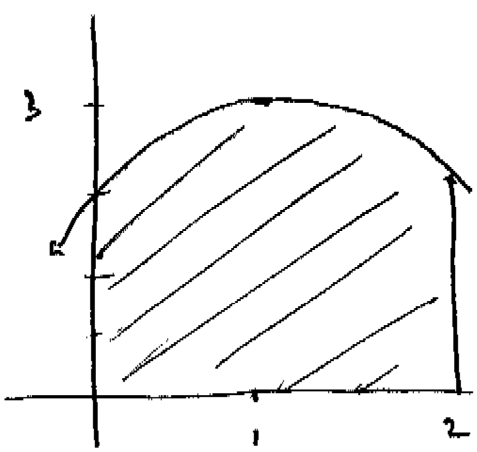
since $e^{rt} > 0 \Rightarrow r^2 + 3r - 4 = 0$

$\Rightarrow (r+4)(r-1) = 0$

$\Rightarrow r = -4$ or $r = 1$

Solution: $r = -4$ or $r = 1$.

5.) (10 pts) Find the centroid of the region bounded by $y = 3-(x-1)^2 = 2+2x-x^2$ on the interval $0 \le x \le 2$ and above the x-axis.



$\bar{x} = 1.$

$$\bar{y} = \frac{1}{A}\int_0^2 \frac{1}{2}(2+2x-x^2)^2\,dx$$

$$= \frac{1}{2A}\int_0^2 (\cancel{4} + 4x - \cancel{2x^2} + 4x + \cancel{4x^2} - 2x^3 - \cancel{2x^2} - 2x^3 + x^4)\,dx$$

$$= \frac{1}{2A}\int_0^2 (4 + 8x - 4x^3 + x^4)\,dx$$

$$= \frac{1}{2A}\left[4x + 4x^2 - x^4 + \frac{x^5}{5}\right]_0^2$$

$$= \frac{1}{2A}\left[8 + \cancel{16} - \cancel{16} + \frac{32}{5}\right] = \frac{1}{2A}\left(\frac{40+32}{5}\right)$$

$$= \frac{72}{10A}$$

$$= \frac{216}{\cancel{360}\,160} = \frac{108}{\cancel{180}\,80} = \frac{54}{\cancel{80}\,40} = \frac{27}{\cancel{20}}$$

Solution: Centroid: $(1, 27/20)$

$y = 3 - (x^2 - 2x + 1)$
$= 3 - x^2 + 2x - 1$
$= 2 + 2x - x^2$

$\Rightarrow A = \int_0^2 (2 + 2x - x^2)\,dx$

$= 2x + x^2 - \frac{x^3}{3}\Big|_0^2$

$= 4 + 4 - \frac{8}{3} \Rightarrow A = \frac{16}{3}$

6.) (10 pts) Derive the circumference of a circle with radius r by finding the arclength of the appropriate function and using symmetry (where necessary).

2 * arclength of semicircle

5/10 $f(x) = (r^2 - x^2)^{1/2}$

$f'(x) = \frac{1}{2}(r^2 - x^2)^{-1/2}(-2x)$

$= \frac{-x}{\sqrt{r^2 - x^2}}$

7/10 $C = 2\int_{-r}^{r} \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx$

$\Rightarrow 4\int_0^r \sqrt{\frac{r^2}{r^2 - x^2}}\, dx$ $x = r\sin\theta$, $dx = r\cos\theta\, d\theta$

$= 4\int_0^{\pi/2} \frac{r}{\sqrt{r^2 - r^2\sin^2\theta}} \cdot r\cos\theta\, d\theta$ $r^2(1-\sin^2\theta) = r^2\cos\theta.$

$= 4\int_0^{\pi/2} r\, d\theta$

$= 4\left[r\theta\right]_0^{\pi/2}$

$= 4r \cdot \pi/2$

Solution: $2\pi r$.

7.) (10 pts) How large do we have to choose n so that the approximation M_n is an accurate approximation to $\int_1^{\pi} \frac{\cos(x)}{x}\, dx$ within 0.00001? The derivatives of the integrand are:

- $f'(x) = -\frac{x\sin x + \cos x}{x^2}$
- $f''(x) = \frac{2x\sin x - (x^2 - 2)\cos x}{x^3}$ max @ $x = 1$ $k = \frac{2\sin(1) + \cos 1}{1}$
- $f^{(3)}(x) = \frac{3(x^2-2)\cos x + x(x^2-6)\sin x}{x^4}$ $\approx 2,223$

$|E_m| \leq \frac{k(\pi - 1)^3}{24 n^2} \leq 0,00001$

$\Rightarrow \frac{2,223(\pi-1)^3}{24\,(0,00001)} \leq n^2 \longrightarrow \sqrt{\frac{2,223(\pi-1)^3}{24\,(0.00001)}} \leq n \leq 302$

$\frac{6}{10}$ w/ random k.

7/10 for $n \geq 102$

Solution: $n \geq 302$.

8.) (10 pts) Complete one of the follow two problems (That is, either (a.) or (b.)).

a.) State the Fundamental Theorem of Calculus, Part 1. Make sure to include all of the conditions and the full conclusion.

If f is cont on $[a,b]$, then $g(x) = \int_a^x f(t)\,dt$ on $a \le x \le b$ is cont on $[a,b]$ and diff on (a,b), and $g' = f$.

b.) State the Fundamental Theorem of Calculus, Part 2. Make sure to include all of the conditions and the full conclusion.

If f is cont on $[a,b]$, then

$\int_a^b f(x)\,dx = F(b) - F(a)$ where $F' = f$.

9.) (10 pts) Find the solution of the differential equation $\frac{dy}{dt} = (t+1)\cdot e^y$ that satisfies the initial condition $y(0) = 0$.

$\Rightarrow \frac{dy}{e^y} = (t+1)\,dt$

$\Rightarrow \int e^{-y}\,dy = \int (t+1)\,dt$

$\Rightarrow -e^{-y} = \frac{t^2}{2} + t + c$

$\Rightarrow e^{-y} = \frac{-t^2}{2} - t + c$

$\Rightarrow y = -\ln\left(\frac{-t^2}{2} - t + c\right)$

$y(0) = 0 = -\ln(c)$

$\Rightarrow c = 1$

3pts

$y = \ln\left(\frac{t^2}{2} + t + 1\right)$

6/10

Solution: $y = -\ln\left(1 - t - t^2/2\right)$

10.) (10 pts) Mom is planning to celebrate the end of the school year with a big turkey feed. The thoroughly defrosted $35°F$ turkey is placed in the preheated oven which is a stable $325°F$. When first placed in the oven, the turkey temperature changes at $0.75\frac{°F}{\text{min}}$. What time should mom put the turkey into the oven so that it will be ready to eat at 4:30pm given that Betty Crocker told her that the turkey will be done when its internal temperature reaches $180°F$.

t = # of min since the turkey went in.

T(t) = turkey temp @ time t.

$$\frac{dT}{dt} = k(T-325)$$

$$\Rightarrow \frac{dT}{T-325} = k\,dt$$

$$\Rightarrow \ln|T-325| = kt + c$$

$$\Rightarrow |\underbrace{T-325}_{\text{negative}}| = \tilde{k}e^{kt}$$

$$\Rightarrow T-325 = -\tilde{k}e^{kt}$$

$$\Rightarrow T = 325 - \tilde{k}e^{kt}$$

initial condition: $T(0) = 35$

$$35 = 325 - \tilde{k} \Rightarrow \tilde{k} = 290.$$

$$\Rightarrow T(t) = 325 - 290e^{kt}$$

$$T'(t) = -290k\,e^{kt}$$

$$T'(0) = -290k = \frac{3}{4}$$

$$\Rightarrow k = -\frac{3}{4(290)} \approx -0.00259$$

$$T(t) = 325 - 290e^{-0.0025862t}$$

Solve

$$180 = 325 - 290e^{-0.0025862t}$$

$$\Rightarrow \frac{1}{2} = e^{-0.0025862t}$$

$$\Rightarrow t = \frac{\ln(1/2)}{-0.0025862}$$

$$\Rightarrow t \approx 268 \text{ min}$$

4:28 time

Solution: Put the turkey in @ 12:02pm