15 SAMPLE EXAM

Problems marked with an asterisk (*) are particularly challenging and should be given careful consideration.

1. Consider the function $f(x, y) = x^y$ on the rectangle $[1, 2] \times [1, 2]$.

(a) Approximate the value of the integral $\int_1^2 \int_1^2 f(x, y)\, dy\, dx$ by dividing the region into four squares and using the function value at the lower left-hand corner of each square as an approximation for the function value over that square.

(b) Does the approximation give an overestimate or an underestimate of the value of the integral? How do you know?

2. Given that $\int_0^{\pi/2} \dfrac{dx}{1+\sin^2 x} = \dfrac{\pi}{2\sqrt{2}}$,

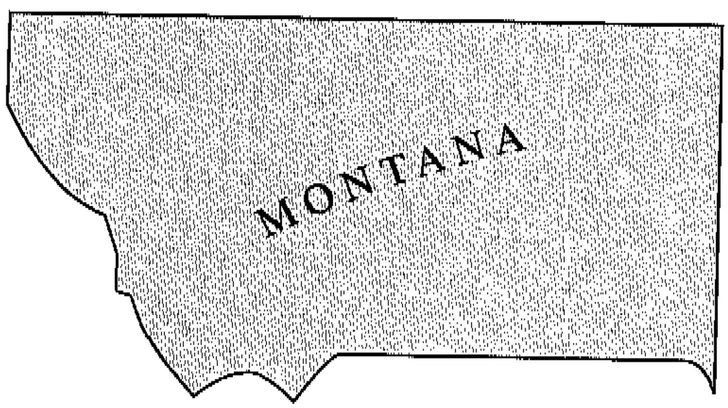
(a) evaluate the double integral

$$\int_0^{\pi/2} \int_0^{\pi/2} \frac{1}{\left(1+\sin^2 x\right)\left(1+\sin^2 y\right)}\, dx\, dy$$

(b) evaluate the triple integral

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_{1/(1+\sin^2 x)}^{1/(1+\sin^2 y)} dz\, dx\, dy$$

3. Consider the rugged region below:



(a) Divide the region into smaller regions, all of which are Type I.

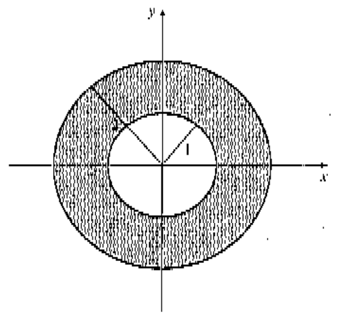
(b) Divide the region into smaller regions, all of which are Type II.

4. Rewrite the integral

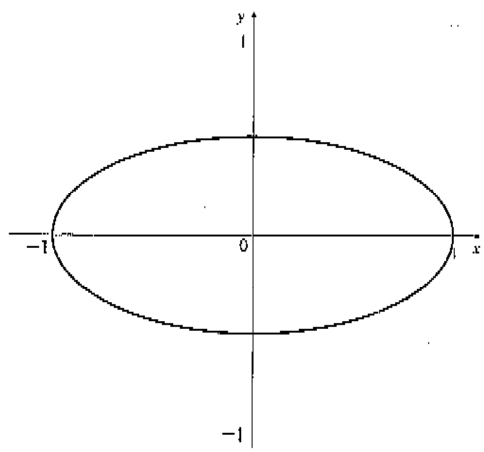
$$\int_0^{2\pi} \int_0^1 r^2\, dr\, d\theta$$

in rectangular coordinates.

5. Evaluate $\iint_D \cos\left(x^2 + y^2\right) dA$, where $D = \left\{(x, y) \mid 1 \le x^2 + y^2 \le 4\right\}$ is a washer with inner radius 1 and outer radius 2.



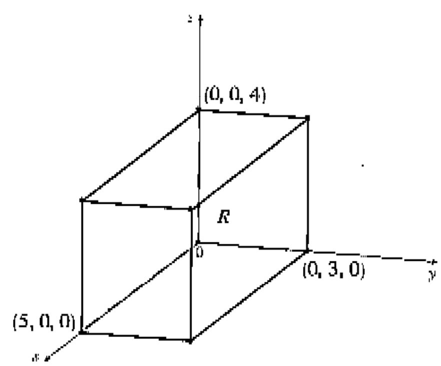
6. Consider the ellipse $x^2 + 2y^2 = 1$.



(a) Rewrite this equation in polar coordinates.

(b) Write an integral in polar coordinates that gives the area of this ellipse. *Note:* Your answer will not look simple.

7. Consider the rectangular prism R pictured below:



Compute $\iiint_R 10\, dV$ and $\iiint_R x\, dV$.

8. (a) Compute

$$\int_0^1 \int_{-1}^1 \int_0^{xy} 1\, dz\, dx\, dy$$

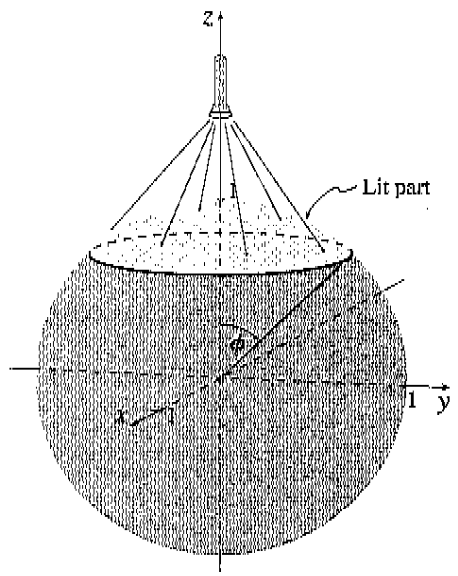
and give a geometric interpretation of your answer.

(b) Compute

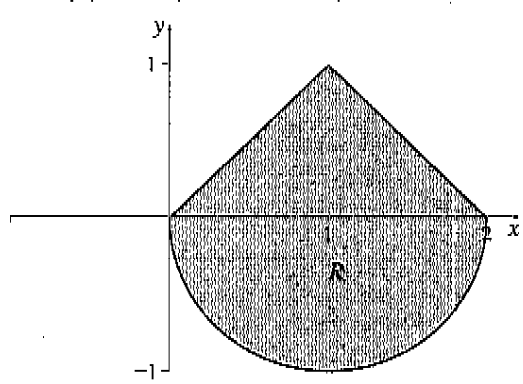
$$\int_0^1 \int_{-1}^1 \int_0^{|xy|} 1\, dz\, dx\, dy$$

and give a geometric interpretation of your answer.

9. A light on the z-axis, pointed at the origin, shines on the sphere $\rho = 1$ such that $\frac{1}{4}$ of the total surface area is lit. What is the angle ϕ?



10. Consider the region R enclosed by $y = x$, $y = -x + 2$, $y = -\sqrt{1 - (x - 1)^2}$:

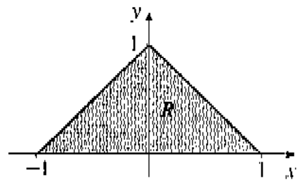


Set up the following integrals as one or more iterated integrals, but do not actually compute them:

(a) $\iint_R (x + y)\, dy\, dx$

(b) $\iint_R (x + y)\, dx\, dy$

11. Consider the region R enclosed by $y = x + 1$, $y = -x + 1$, and the x-axis.



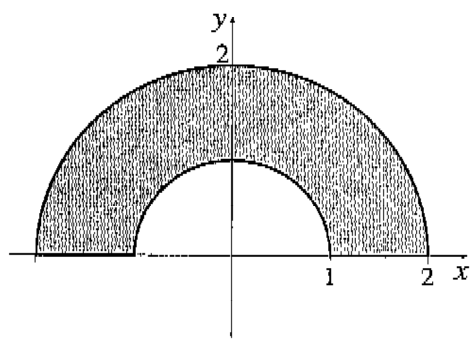
(a) Set up the integral $\iint_R xy\,dx\,dy$ in polar coordinates.

(b) Compute the integral $\iint_R xy\,dx\,dy$ using any method you know.

12. Consider the double integral

$$\iint_R \frac{1}{9 - (x^2 + y^2)^{3/2}}\,dA$$

where R is given by the region between the two semicircles pictured below:

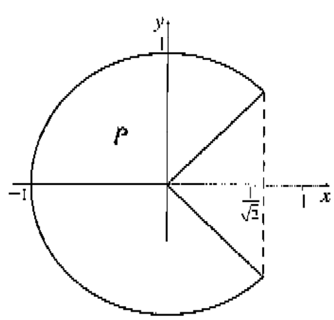


(a) Compute the shaded area.

(b) Show that the function $\dfrac{1}{9 - (x^2 + y^2)^{3/2}}$ is constant on each of the two bounding semicircles.

(c) Give a lower bound and an upper bound for the double integral using the above information.

13. Observe the following Pac-Man:



(a) Describe him in polar coordinates.

(b) Evaluate $\iint_{\text{Pac-Man}} x\,dA$ and $\iint_{\text{Pac-Man}} y\,dA$.

14. Consider the triple integral

$$\int_0^1 \int_{y^3}^{\sqrt{y}} \int_0^{xy} dz\,dx\,dy$$

representing a solid S. Let R be the projection of S onto the plane $z = 0$.

(a) Draw the region R.

(b) Rewrite this integral as $\iiint_S dz\,dy\,dx$.

15. Consider the transformation T: $x = 2u + v$, $y = u + 2v$.

(a) Describe the image S under T of the unit square $R = \{(u, v) \mid 0 \leq u \leq 1, 0 \leq v \leq 1\}$ in the uv-plane using a change of coordinates.

(b) Evaluate $\iint_S (3x + 2y)\,dA$

16. What is the volume of the following region, described in spherical coordinates: $1 \leq \rho \leq 9$, $0 \leq \theta \leq \frac{\pi}{2}$, $\frac{\pi}{6} \leq \phi \leq \frac{\pi}{4}$?

17. Consider the transformation $x = v \cos 2\pi u$, $y = v \sin 2\pi u$.

(a) Describe the image S under T of the unit square $R = \{(u, v) \mid 0 \leq u \leq 1, 0 \leq v \leq 1\}$.

(b) Find the area of S.

18. Consider the function $f(x, y) = ax + by$, where a and b are constants. Find the average value of f over the region $R = \{(x, y) \mid -1 \leq x \leq 1, -1 \leq y \leq 1\}$.