CONCEPT CHECK

1. See Definitions 1 and 2 in Section 17.1 [ET 16.1]. A vector field can represent, for example, the wind velocity at any location in space, the speed and direction of the ocean current at any location, or the force vectors of Earth's gravitational field at a location in space.

2. (a) A conservative vector field $\mathbf{F}$ is a vector field which is the gradient of some scalar function f.

   (b) The function f in part (a) is called a potential function for $\mathbf{F}$, that is, $\mathbf{F} = \nabla f$.

3. (a) See Definition 17.2.2 [ET 16.2.2].

   (b) We normally evaluate the line integral using Formula 17.2.3 [ET 16.2.3].

   (c) The mass is $m = \int_C \rho(x,y)\, ds$, and the center of mass is $(\overline{x}, \overline{y})$ where $\overline{x} = \frac{1}{m}\int_C x\rho(x,y)\, ds$, $\overline{y} = \frac{1}{m}\int_C y\rho(x,y)\, ds$.

   (d) See (5) and (6) in Section 17.2 [ET 16.2] for plane curves; we have similar definitions when C is a space curve (see the equation preceding (10) in Section 17.2 [ET 16.2]).

   (e) For plane curves, see Equations 17.2.7 [ET 16.2.7]. We have similar results for space curves (see the equation preceding (10) in Section 17.2 [ET 16.2]).

4. (a) See Definition 17.2.13 [ET 16.2.13].

   (b) If $\mathbf{F}$ is a force field, $\int_C \mathbf{F}\cdot d\mathbf{r}$ represents the work done by $\mathbf{F}$ in moving a particle along the curve C.

   (c) $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz$

5. See Theorem 17.3.2 [ET 16.3.2].

6. (a) $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path if the line integral has the same value for any two curves that have the same initial and terminal points.

   (b) See Theorem 17.3.4 [ET 16.3.4].

7. See the statement of Green's Theorem on page 1091 [ET 1055].

8. See Equations 17.4.5 [ET 16.4.5].

9. (a) $\operatorname{curl}\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k} = \nabla\times\mathbf{F}$

   (b) $\operatorname{div}\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla\cdot\mathbf{F}$

   (c) For curl $\mathbf{F}$, see the discussion accompanying Figure 1 on page 1100 [ET 1064] as well as Figure 6 and the accompanying discussion on page 1132 [ET 1096]. For div $\mathbf{F}$, see the discussion following Example 5 on page 1102 [ET 1066] as well as the discussion preceding (8) on page 1139 [ET 1103].

10. See Theorem 17.3.6 [ET 16.3.6]; see Theorem 17.5.4 [ET 16.5.4].

11. (a) See (1) and (2) and the accompanying discussion in Section 17.6 [ET 16.6] ; See Figure 4 and the accompanying discussion on page 1107 [ET 1071] .

(b) See Definition 17.6.6 [ET 16.6.6].

(c) See Equation 17.6.9 [ET 16.6.9].

12. (a) See (1) in Section 17.7 [ET 16.7].

(b) We normally evaluate the surface integral using Formula 17.7.2 [ET 16.7.2].

(c) See Formula 17.7.4 [ET 16.7.4].

(d) The mass is $m = \iint_S \rho(x, y, z)\, dS$ and the center of mass is $(\overline{x}, \overline{y}, \overline{z})$ where $\overline{x} = \frac{1}{m} \iint_S x\rho(x, y, z)\, dS$, $\overline{y} = \frac{1}{m} \iint_S y\rho(x, y, z)\, dS$, $\overline{z} = \frac{1}{m} \iint_S z\rho(x, y, z)\, dS$.

13. (a) See Figures 6 and 7 and the accompanying discussion in Section 17.7 [ET 16.7]. A Möbius strip is a nonorientable surface; see Figures 4 and 5 and the accompanying discussion on page 1121 [ET 1085].

(b) See Definition 17.7.8 [ET 16.7.8].

(c) See Formula 17.7.9 [ET 16.7.9].

(d) See Formula 17.7.10 [ET 16.7.10].

14. See the statement of Stokes' Theorem on page 1129 [ET 1093.].

15. See the statement of the Divergence Theorem on page 1135 [ET 1099].

16. In each theorem, we have an integral of a "derivative" over a region on the left side, while the right side involves the values of the original function only on the boundary of the region.

TRUE-FALSE QUIZ

1. False; div $\mathbf{F}$ is a scalar field.

2. True. (See Definition 17.5.1 [ET 16.5.1].)

3. True, by Theorem 17.5.3 [ET 16.5.3] and the fact that div $\mathbf{0} = 0$.

4. True, by Theorem 17.3.2 [ET 16.3.2].

5. False. See Exercise 17.3.33 [ET 16.3.33]. (But the assertion is true if D is simply-connected; see Theorem 17.3.6 [ET 16.3.6].)

6. False. See the discussion accompanying Figure 8 on page 1075 [ET 1039].

7. True. Apply the Divergence Theorem and use the fact that div $\mathbf{F} = 0$.

8. False by Theorem 17.5.11 [ET 16.5.11], because if it were true, then div curl $\mathbf{F} = 3 \neq 0$.

EXERCISES

1. (a) Vectors starting on C point in roughly the direction opposite to C, so the tangential component $\mathbf{F}\cdot\mathbf{T}$ is negative. Thus $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$ is negative.

   (b) The vectors that end near P are shorter than the vectors that start near P, so the net flow is outward near P and $\operatorname{div}\mathbf{F}(P)$ is positive.

2. We can parametrize C by $x = x$, $y = x^2$, $0 \le x \le 1$ so

$$\int_C x\,ds = \int_0^1 x\sqrt{1+(2x)^2}\,dx = \tfrac{1}{12}(1+4x^2)^{3/2}\Big]_0^1 = \tfrac{1}{12}(5\sqrt{5}-1).$$

3. $$\int_C yz\cos x\,ds = \int_0^\pi (3\cos t)(3\sin t)\cos t\sqrt{(1)^2+(-3\sin t)^2+(3\cos t)^2}\,dt = \int_0^\pi (9\cos^2 t\,\sin t)\sqrt{10}\,dt$$
$$= 9\sqrt{10}\left(-\tfrac{1}{3}\cos^3 t\right)\Big]_0^\pi = -3\sqrt{10}\,(-2) = 6\sqrt{10}$$

4. $x = 3\cos t \;\Rightarrow\; dx = -3\sin t\,dt$, $y = 2\sin t \;\Rightarrow\; dy = 2\cos t\,dt$, $0 \le t \le 2\pi$, so

$$\int_C y\,dx + (x+y^2)\,dy = \int_0^{2\pi}\left[(2\sin t)(-3\sin t) + (3\cos t + 4\sin^2 t)(2\cos t)\right]dt$$
$$= \int_0^{2\pi}(-6\sin^2 t + 6\cos^2 t + 8\sin^2 t\,\cos t)\,dt = \int_0^{2\pi}\left[6(\cos^2 t - \sin^2 t) + 8\sin^2 t\,\cos t\right]dt$$
$$= \int_0^{2\pi}(6\cos 2t + 8\sin^2 t\,\cos t)\,dt = 3\sin 2t + \tfrac{8}{3}\sin^3 t\Big]_0^{2\pi} = 0$$

Or: Notice that $\frac{\partial}{\partial y}(y) = 1 = \frac{\partial}{\partial x}(x+y^2)$, so $\mathbf{F}(x,y) = \langle y, x+y^2\rangle$ is a conservative vector field. Since C is a closed curve, $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C y\,dx + (x+y^2)\,dy = 0$.

5. $$\int_C y^3\,dx + x^3\,dy = \int_{-1}^1\left[y^3(-2y) + (1-y^2)^3\right]dy = \int_{-1}^1(-y^4 - 2y^2 + 1)\,dy$$
$$= \left[-\tfrac{1}{5}y^5 - \tfrac{2}{3}y^3 + y\right]_{-1}^1 = -\tfrac{1}{5} - \tfrac{2}{3} + 1 - \tfrac{1}{5} - \tfrac{2}{3} + 1 = \tfrac{4}{15}$$

6. $$\int_C \sqrt{xy}\,dx + e^y\,dy + xz\,dz = \int_0^1\left(\sqrt{t^4\cdot t^2}\cdot 4t^3 + e^{t^2}\cdot 2t + t^4\cdot t^3\cdot 3t^2\right)dt = \int_0^1(4t^6 + 2te^{t^2} + 3t^9)\,dt$$
$$= \left[\tfrac{4}{7}t^7 + e^{t^2} + \tfrac{3}{10}t^{10}\right]_0^1 = e - \tfrac{9}{70}$$

7. C: $x = 1+2t \;\Rightarrow\; dx = 2\,dt$, $y = 4t \;\Rightarrow\; dy = 4\,dt$, $z = -1+3t \;\Rightarrow\; dz = 3\,dt$, $0 \le t \le 1$.

$$\int_C xy\,dx + y^2\,dy + yz\,dz = \int_0^1[(1+2t)(4t)(2) + (4t)^2(4) + (4t)(-1+3t)(3)]\,dt$$
$$= \int_0^1(116t^2 - 4t)\,dt = \left[\tfrac{116}{3}t^3 - 2t^2\right]_0^1 = \tfrac{116}{3} - 2 = \tfrac{110}{3}$$

8. $\mathbf{F}(\mathbf{r}(t)) = (\sin t)(1+t)\,\mathbf{i} + (\sin^2 t)\,\mathbf{j}$, $\mathbf{r}'(t) = \cos t\,\mathbf{i} + \mathbf{j}$ and

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^\pi((1+t)\sin t\cos t + \sin^2 t)\,dt = \int_0^\pi\left(\tfrac{1}{2}(1+t)\sin 2t + \sin^2 t\right)dt$$
$$= \left[\tfrac{1}{2}\left((1+t)\left(-\tfrac{1}{2}\cos 2t\right) + \tfrac{1}{4}\sin 2t\right) + \tfrac{1}{2}t - \tfrac{1}{4}\sin 2t\right]_0^\pi = \tfrac{\pi}{4}$$

9. $\mathbf{F}(\mathbf{r}(t)) = e^{-t}\,\mathbf{i} + t^2(-t)\,\mathbf{j} + (t^2+t^3)\,\mathbf{k}$, $\mathbf{r}'(t) = 2t\,\mathbf{i} + 3t^2\,\mathbf{j} - \mathbf{k}$ and

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1(2te^{-t} - 3t^5 - (t^2+t^3))\,dt = \left[-2te^{-t} - 2e^{-t} - \tfrac{1}{2}t^6 - \tfrac{1}{3}t^3 - \tfrac{1}{4}t^4\right]_0^1 = \tfrac{11}{12} - \tfrac{4}{e}.$$

10. (a) C: $x = 3 - 3t$, $y = \frac{\pi}{2}t$, $z = 3t$, $0 \le t \le 1$. Then

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \left[3t\,\mathbf{i} + (3 - 3t)\,\mathbf{j} + \tfrac{\pi}{2}t\,\mathbf{k}\right] \cdot \left[-3\,\mathbf{i} + \tfrac{\pi}{2}\,\mathbf{j} + 3\,\mathbf{k}\right] dt = \int_0^1 \left[-9t + \tfrac{3\pi}{2}\right] dt = \tfrac{1}{2}(3\pi - 9).$$

(b) $$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} (3 \sin t\,\mathbf{i} + 3 \cos t\,\mathbf{j} + t\,\mathbf{k}) \cdot (-3 \sin t\,\mathbf{i} + \mathbf{j} + 3 \cos t\,\mathbf{k})\,dt$$
$$= \int_0^{\pi/2} (-9 \sin^2 t + 3 \cos t + 3t \cos t)\,dt = \left[-\tfrac{9}{2}(t - \sin t \cos t) + 3 \sin t + 3(t \sin t + \cos t)\right]_0^{\pi/2}$$
$$= -\tfrac{9\pi}{4} + 3 + \tfrac{3\pi}{2} - 3 = -\tfrac{3\pi}{4}$$

11. $\frac{\partial}{\partial y}\left[(1 + xy)e^{xy}\right] = 2xe^{xy} + x^2ye^{xy} = \frac{\partial}{\partial x}\left[e^y + x^2e^{xy}\right]$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$, so $\mathbf{F}$ is conservative. Thus there exists a function f such that $\mathbf{F} = \nabla f$. Then $f_y(x, y) = e^y + x^2e^{xy}$ implies $f(x, y) = e^y + xe^{xy} + g(x)$ and then $f_x(x, y) = xye^{xy} + e^{xy} + g'(x) = (1 + xy)e^{xy} + g'(x)$. But $f_x(x, y) = (1 + xy)e^{xy}$, so $g'(x) = 0 \Rightarrow g(x) = K$. Thus $f(x, y) = e^y + xe^{xy} + K$ is a potential function for $\mathbf{F}$.

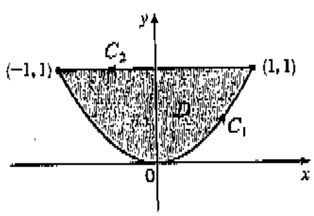
12. $\mathbf{F}$ is defined on all of $\mathbb{R}^3$, its components have continuous partial derivatives, and $\operatorname{curl} \mathbf{F} = (0 - 0)\,\mathbf{i} - (0 - 0)\,\mathbf{j} + (\cos y - \cos y)\,\mathbf{k} = \mathbf{0}$, so $\mathbf{F}$ is conservative by Theorem 17.5.4 [ET 16.5.4]. Thus there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x, y, z) = \sin y$ implies $f(x, y, z) = x \sin y + g(y, z)$ and then $f_y(x, y, z) = x \cos y + g_y(y, z)$. But $f_y(x, y, z) = x \cos y$, so $g_y(y, z) = 0 \Rightarrow g(y, z) = h(z)$. Then $f(x, y, z) = x \sin y + h(z)$ implies $f_z(x, y, z) = h'(z)$. But $f_z(x, y, z) = -\sin z$, so $h(z) = \cos z + K$. Thus a potential function for $\mathbf{F}$ is $f(x, y, z) = x \sin y + \cos z + K$.

13. Since $\frac{\partial}{\partial y}(4x^3y^2 - 2xy^3) = 8x^3y - 6xy^2 = \frac{\partial}{\partial x}(2x^4y - 3x^2y^2 + 4y^3)$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$, $\mathbf{F}$ is conservative. Furthermore $f(x, y) = x^4y^2 - x^2y^3 + y^4$ is a potential function for $\mathbf{F}$. $t = 0$ corresponds to the point $(0, 1)$ and $t = 1$ corresponds to $(1, 1)$, so $\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 1) - f(0, 1) = 1 - 1 = 0$.

14. Here $\operatorname{curl} \mathbf{F} = \mathbf{0}$, the domain of $\mathbf{F}$ is $\mathbb{R}^3$, and the components of $\mathbf{F}$ have continuous partial derivatives, so $\mathbf{F}$ is conservative. Furthermore $f(x, y, z) = xe^y + ye^z$ is a potential function for $\mathbf{F}$. Then $\int_C \mathbf{F} \cdot d\mathbf{r} = f(4, 0, 3) - f(0, 2, 0) = 4 - 2 = 2$.

15. C_1: $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j}$, $-1 \le t \le 1$;

C_2: $\mathbf{r}(t) = -t\,\mathbf{i} + \mathbf{j}$, $-1 \le t \le 1$.



Then

$$\int_C xy^2\,dx - x^2y\,dy = \int_{-1}^{1} (t^5 - 2t^5)\,dt + \int_{-1}^{1} t\,dt = \left[-\tfrac{1}{6}t^6\right]_{-1}^{1} + \left[\tfrac{1}{2}t^2\right]_{-1}^{1} = 0$$

Using Green's Theorem, we have

$$\int_C xy^2\,dx - x^2y\,dy = \iint_D \left[\frac{\partial}{\partial x}(-x^2y) - \frac{\partial}{\partial y}(xy^2)\right] dA = \iint_D (-2xy - 2xy)\,dA = \int_{-1}^{1} \int_{x^2}^{1} -4xy\,dy\,dx$$
$$= \int_{-1}^{1} \left[-2xy^2\right]_{y=x^2}^{y=1} dx = \int_{-1}^{1} (2x^5 - 2x)\,dx = \left[\tfrac{1}{3}x^6 - x^2\right]_{-1}^{1} = 0$$

16. $\int_C \sqrt{1 + x^3}\,dx + 2xy\,dy = \iint_D \left[\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(\sqrt{1 + x^3})\right] dA = \int_0^1 \int_0^{3x} (2y - 0)\,dy\,dx = \int_0^1 9x^2\,dx = 3x^3\Big]_0^1 = 3$

17. $\int_C x^2y\,dx - xy^2\,dy = \iint_{x^2 + y^2 \le 4} \left[\frac{\partial}{\partial x}(-xy^2) - \frac{\partial}{\partial y}(x^2y)\right] dA = \iint_{x^2 + y^2 \le 4} (-y^2 - x^2)\,dA = -\int_0^{2\pi} \int_0^2 r^3\,dr\,d\theta = -8\pi$

18. $\operatorname{curl}\mathbf{F} = (0 - e^{-y}\cos z)\,\mathbf{i} - (e^{-z}\cos x - 0)\,\mathbf{j} + (0 - e^{-x}\cos y)\,\mathbf{k} = -e^{-y}\cos z\,\mathbf{i} - e^{-z}\cos x\,\mathbf{j} - e^{-x}\cos y\,\mathbf{k}$,

$\operatorname{div}\mathbf{F} = -e^{-x}\sin y - e^{-y}\sin z - e^{-z}\sin x$

19. If we assume there is such a vector field $\mathbf{G}$, then $\operatorname{div}(\operatorname{curl}\mathbf{G}) = 2 + 3z - 2xz$. But $\operatorname{div}(\operatorname{curl}\mathbf{F}) = 0$ for all vector fields $\mathbf{F}$. Thus such a $\mathbf{G}$ cannot exist.

20. Let $\mathbf{F} = P_1\,\mathbf{i} + Q_1\,\mathbf{j} + R_1\,\mathbf{k}$ and $\mathbf{G} = P_2\,\mathbf{i} + Q_2\,\mathbf{j} + R_2\,\mathbf{k}$ be vector fields whose first partials exist and are continuous. Then

$$\begin{aligned}\mathbf{F}\operatorname{div}\mathbf{G} - \mathbf{G}\operatorname{div}\mathbf{F} &= \left[P_1\left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z}\right)\mathbf{i} + Q_1\left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z}\right)\mathbf{j} + R_1\left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z}\right)\mathbf{k}\right]\\ &\quad - \left[P_2\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right)\mathbf{i} + Q_2\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right)\mathbf{j}\right.\\ &\quad\left. + R_2\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R_1}{\partial z}\right)\mathbf{k}\right]\end{aligned}$$

and

$$\begin{aligned}(\mathbf{G}\cdot\nabla)\,\mathbf{F} - (\mathbf{F}\cdot\nabla)\,\mathbf{G} &= \left[\left(P_2\frac{\partial P_1}{\partial x} + Q_2\frac{\partial P_1}{\partial y} + R_2\frac{\partial P_1}{\partial z}\right)\mathbf{i} + \left(P_2\frac{\partial Q_1}{\partial x} + Q_2\frac{\partial Q_1}{\partial y} + R_2\frac{\partial Q_1}{\partial z}\right)\mathbf{j}\right.\\ &\quad\left. + \left(P_2\frac{\partial R_1}{\partial x} + Q_2\frac{\partial R_1}{\partial y} + R_2\frac{\partial R_1}{\partial z}\right)\mathbf{k}\right]\\ &\quad - \left[\left(P_1\frac{\partial P_2}{\partial x} + Q_1\frac{\partial P_2}{\partial y} + R_1\frac{\partial P_2}{\partial z}\right)\mathbf{i} + \left(P_1\frac{\partial Q_2}{\partial x} + Q_1\frac{\partial Q_2}{\partial y} + R_1\frac{\partial Q_2}{\partial z}\right)\mathbf{j}\right.\\ &\quad\left. + \left(P_1\frac{\partial R_2}{\partial x} + Q_1\frac{\partial R_2}{\partial y} + R_1\frac{\partial R_2}{\partial z}\right)\mathbf{k}\right]\end{aligned}$$

Hence

$$\begin{aligned}&\mathbf{F}\operatorname{div}\mathbf{G} - \mathbf{G}\operatorname{div}\mathbf{F} + (\mathbf{G}\cdot\nabla)\,\mathbf{F} - (\mathbf{F}\cdot\nabla)\,\mathbf{G}\\ &\quad= \left[\left(P_1\frac{\partial Q_2}{\partial y} + Q_2\frac{\partial P_1}{\partial x}\right) - \left(P_2\frac{\partial Q_1}{\partial y} + Q_1\frac{\partial P_2}{\partial y}\right)\right.\\ &\qquad\left. - \left(P_2\frac{\partial R_1}{\partial z} + R_1\frac{\partial P_2}{\partial z}\right) + \left(P_1\frac{\partial R_2}{\partial z} + R_2\frac{\partial P_1}{\partial z}\right)\right]\mathbf{i}\\ &\qquad + \left[\left(Q_1\frac{\partial R_2}{\partial z} + R_2\frac{\partial Q_1}{\partial z}\right) - \left(Q_2\frac{\partial R_1}{\partial z} + R_1\frac{\partial Q_2}{\partial z}\right)\right.\\ &\qquad\left. - \left(P_1\frac{\partial Q_2}{\partial x} + Q_2\frac{\partial P_1}{\partial x}\right) + \left(P_2\frac{\partial Q_1}{\partial x} + Q_1\frac{\partial P_2}{\partial x}\right)\right]\mathbf{j}\\ &\qquad + \left[\left(P_2\frac{\partial R_1}{\partial x} + R_1\frac{\partial P_2}{\partial x}\right) - \left(P_1\frac{\partial R_2}{\partial x} + R_2\frac{\partial P_1}{\partial x}\right)\right.\\ &\qquad\left. - \left(Q_1\frac{\partial R_2}{\partial y} + R_2\frac{\partial Q_1}{\partial y}\right) + \left(Q_2\frac{\partial R_1}{\partial y} + R_1\frac{\partial Q_2}{\partial y}\right)\right]\mathbf{k}\\ &\quad= \left[\frac{\partial}{\partial y}(P_1Q_2 - P_2Q_1) - \frac{\partial}{\partial z}(P_2R_1 - P_1R_2)\right]\mathbf{i}\\ &\qquad + \left[\frac{\partial}{\partial z}(Q_1R_2 - Q_2R_1) - \frac{\partial}{\partial x}(P_1Q_2 - P_2Q_1)\right]\mathbf{j}\\ &\qquad + \left[\frac{\partial}{\partial x}(P_2R_1 - P_1R_2) - \frac{\partial}{\partial y}(Q_1R_2 - Q_2R_1)\right]\mathbf{k}\\ &\quad= \operatorname{curl}(\mathbf{F}\times\mathbf{G})\end{aligned}$$

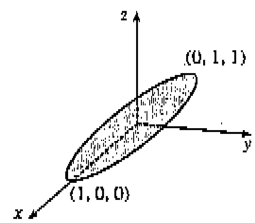
21. For any piecewise-smooth simple closed plane curve C bounding a region D, we can apply Green's Theorem to $\mathbf{F}(x,y) = f(x)\,\mathbf{i} + g(y)\,\mathbf{j}$ to get $\int_C f(x)\,dx + g(y)\,dy = \iint_D \left[\frac{\partial}{\partial x} g(y) - \frac{\partial}{\partial y} f(x)\right] dA = \iint_D 0\,dA = 0$.

22.
$$\begin{aligned}\nabla^2(fg) &= \frac{\partial^2(fg)}{\partial x^2} + \frac{\partial^2(fg)}{\partial y^2} + \frac{\partial^2(fg)}{\partial z^2}\\ &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} g + f\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y} g + f\frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z} g + f\frac{\partial g}{\partial z}\right) \quad \text{[Product Rule]}\\ &= \frac{\partial^2 f}{\partial x^2} g + 2\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + f\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} g + 2\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}\\ &\qquad + f\frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} g + 2\frac{\partial f}{\partial z}\frac{\partial g}{\partial z} + f\frac{\partial^2 g}{\partial z^2} \quad \text{[Product Rule]}\\ &= f\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}\right) + g\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) + 2\left\langle\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right\rangle \cdot \left\langle\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right\rangle\\ &= f\nabla^2 g + g\nabla^2 f + 2\nabla f \cdot \nabla g\end{aligned}$$

Another method: Using the rules in Exercises 15.6.37(b) [ET 14.6.37(b)] and 17.5.25 [ET 16.5.25], we have
$$\begin{aligned}\nabla^2(fg) &= \nabla\cdot\nabla(fg) = \nabla\cdot(g\,\nabla f + f\,\nabla g) = \nabla g\cdot\nabla f + g\nabla\cdot\nabla f + \nabla f\cdot\nabla g + f\nabla\cdot\nabla g\\ &= g\,\nabla^2 f + f\,\nabla^2 g + 2\nabla f\cdot\nabla g\end{aligned}$$

23. $\nabla^2 f = 0$ means that $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$. Now if $\mathbf{F} = f_y\,\mathbf{i} - f_x\,\mathbf{j}$ and C is any closed path in D, then applying Green's Theorem, we get
$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C f_y\,dx - f_x\,dy = \iint_D \left[\frac{\partial}{\partial x}(-f_x) - \frac{\partial}{\partial y}(f_y)\right] dA = -\iint_D (f_{xx} + f_{yy})\,dA = -\iint_D 0\,dA = 0$$
Therefore the line integral is independent of path, by Theorem 17.3.3 [ET 16.3.3].

24. (a) $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, so C lies on the circular cylinder $x^2 + y^2 = 1$. But also $y = z$, so C lies on the plane $y = z$. Thus C is the intersection of the plane $y = z$ and the cylinder $x^2 + y^2 = 1$.



(b) Apply Stokes' Theorem, $\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$:

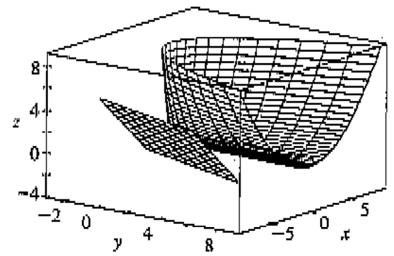
$$\operatorname{curl}\mathbf{F} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ 2xe^{2y} & 2x^2e^{2y} + 2y\cot z & -y^2\csc^2 z\end{vmatrix} = \left\langle -2y\csc^2 z - (-2y\csc^2 z), 0, 4xe^{2y} - 4xe^{2y}\right\rangle = \mathbf{0}$$

Therefore $\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \mathbf{0}\cdot d\mathbf{S} = 0$.

25. $z = f(x,y) = x^2 + 2y$ with $0 \le x \le 1$, $0 \le y \le 2x$. Thus
$$A(S) = \iint_D \sqrt{1 + 4x^2 + 4}\,dA = \int_0^1\int_0^{2x} \sqrt{5 + 4x^2}\,dy\,dx = \int_0^1 2x\sqrt{5 + 4x^2}\,dx = \tfrac{1}{6}(5 + 4x^2)^{3/2}\Big]_0^1 = \tfrac{1}{6}\left(27 - 5\sqrt{5}\right).$$

26. (a) $\mathbf{r}_u = -v\,\mathbf{j} + 2u\,\mathbf{k}$, $\mathbf{r}_v = 2v\,\mathbf{i} - u\,\mathbf{j}$ and $\mathbf{r}_u \times \mathbf{r}_v = 2u^2\,\mathbf{i} + 4uv\,\mathbf{j} + 2v^2\,\mathbf{k}$. Since the point $(4, -2, 1)$ corresponds to $u = 1$, $v = 2$ (or $u = -1$, $v = -2$ but $\mathbf{r}_u \times \mathbf{r}_v$ is the same for both), a normal vector to the surface at $(4, -2, 1)$ is $2\,\mathbf{i} + 8\,\mathbf{j} + 8\,\mathbf{k}$ and an equation of the tangent plane is $2x + 8y + 8z = 0$ or $x + 4y + 4z = 0$.

(b)



(c) By Definition 17.6.6 [ET 16.6.6], the area of S is given by

$$A(S) = \int_0^3 \int_{-3}^3 \sqrt{(2u^2)^2 + (4uv)^2 + (2v^2)^2}\,dv\,du = 2\int_0^3 \int_{-3}^3 \sqrt{u^4 + 4u^2v^2 + v^4}\,dv\,du.$$

(d) By Equation 17.7.9 [ET 16.7.9], the surface integral is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^3 \int_{-3}^3 \left\langle \frac{(u^2)^2}{1 + (v^2)^2}, \frac{(v^2)^2}{1 + (-uv)^2}, \frac{(-uv)^2}{1 + (u^2)^2} \right\rangle \cdot \langle 2u^2, 4uv, 2v^2 \rangle\,dv\,du$$

$$= \int_0^3 \int_{-3}^3 \left(\frac{2u^6}{1 + v^4} + \frac{4uv^5}{1 + u^2v^2} + \frac{2u^2v^4}{1 + u^4} \right) dv\,du \approx 1524.0190$$

27. $z = f(x, y) = x^2 + y^2$ with $0 \le x^2 + y^2 \le 4$ so $\mathbf{r}_x \times \mathbf{r}_y = -2x\,\mathbf{i} - 2y\,\mathbf{j} + \mathbf{k}$ (using upward orientation). Then

$$\iint_S z\,dS = \iint_{x^2 + y^2 \le 4} (x^2 + y^2)\sqrt{4x^2 + 4y^2 + 1}\,dA = \int_0^{2\pi} \int_0^2 r^3 \sqrt{1 + 4r^2}\,dr\,d\theta = \tfrac{1}{60}\pi\left(391\sqrt{17} + 1\right)$$

(Substitute $u = 1 + 4r^2$ and use tables.)

28. $z = f(x, y) = 4 + x + y$ with $0 \le x^2 + y^2 \le 4$ so $\mathbf{r}_x \times \mathbf{r}_y = -\mathbf{i} - \mathbf{j} + \mathbf{k}$. Then

$$\iint_S (x^2 z + y^2 z)\,dS = \iint_{x^2 + y^2 \le 4} (x^2 + y^2)(4 + x + y)\sqrt{3}\,dA$$

$$= \int_0^2 \int_0^{2\pi} \sqrt{3}\,r^3(4 + r\cos\theta + r\sin\theta)\,d\theta\,dr = \int_0^2 8\pi\sqrt{3}\,r^3\,dr = 32\pi\sqrt{3}$$

29. Since the sphere bounds a simple solid region, the Divergence Theorem applies and

$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E (z - 2)\,dV = \iiint_E z\,dV - 2\iiint_E dV = m\bar{z} - 2\left(\frac{4}{3}\pi 2^3\right) = -\frac{64}{3}\pi.$

Alternate solution: $\mathbf{F}(\mathbf{r}(\phi, \theta)) = 4\sin\phi\,\cos\theta\,\cos\phi\,\mathbf{i} - 4\sin\phi\,\sin\theta\,\mathbf{j} + 6\sin\phi\,\cos\theta\,\mathbf{k}$,

$\mathbf{r}_\phi \times \mathbf{r}_\theta = 4\sin^2\phi\,\cos\theta\,\mathbf{i} + 4\sin^2\phi\,\sin\theta\,\mathbf{j} + 4\sin\phi\,\cos\phi\,\mathbf{k}$, and

$\mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) = 16\sin^3\phi\,\cos^2\theta\,\cos\phi - 16\sin^3\phi\,\sin^2\theta + 24\sin^2\phi\,\cos\phi\,\cos\theta$. Then

$$\iint_S F \cdot dS = \int_0^{2\pi} \int_0^{\pi} (16\sin^3\phi\,\cos\phi\,\cos^2\theta - 16\sin^3\phi\,\sin^2\theta + 24\sin^2\phi\,\cos\phi\,\cos\theta)\,d\phi\,d\theta$$

$$= \int_0^{2\pi} \tfrac{4}{3}(-16\sin^2\theta)\,d\theta = -\tfrac{64}{3}\pi$$

30. $z = f(x, y) = x^2 + y^2$, $\mathbf{r}_x \times \mathbf{r}_y = -2x\,\mathbf{i} - 2y\,\mathbf{j} + \mathbf{k}$ (because of upward orientation) and $\mathbf{F}(\mathbf{r}(x, y)) \cdot (\mathbf{r}_x \times \mathbf{r}_y) = -2x^3 - 2xy^2 + x^2 + y^2$. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{x^2 + y^2 \le 1} (-2x^3 - 2xy^2 + x^2 + y^2)\,dA$$

$$= \int_0^1 \int_0^{2\pi} (-2r^3\cos^3\theta - 2r^3\cos\theta\,\sin^2\theta + r^2)\,r\,dr\,d\theta = \int_0^1 r^3(2\pi)\,dr = \tfrac{\pi}{2}$$

31. Since $\operatorname{curl}\mathbf{F} = \mathbf{0}$, $\iint_S (\operatorname{curl}\mathbf{F}) \cdot d\mathbf{S} = 0$. We parametrize C: $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \le t \le 2\pi$ and

$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (-\cos^2 t\,\sin t + \sin^2 t\,\cos t)\,dt = \frac{1}{3}\cos^3 t + \frac{1}{3}\sin^3 t\Big]_0^{2\pi} = 0.$

32. $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \oint_C \mathbf{F}\cdot d\mathbf{r}$ where C: $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + \mathbf{k}$, $0 \le t \le 2\pi$, so $\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$,

$\mathbf{F}(\mathbf{r}(t)) = 8\cos^2 t\,\sin t\,\mathbf{i} + 2\sin t\,\mathbf{j} + e^{4\cos t\sin t}\,\mathbf{k}$, and $\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) = -16\cos^2 t\sin^2 t + 4\sin t\cos t$. Thus

$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}(-16\cos^2 t\,\sin^2 t + 4\sin t\,\cos t)\,dt = \left[-16\left(-\frac{1}{4}\sin t\,\cos^3 t + \frac{1}{16}\sin 2t + \frac{1}{8}t\right) + 2\sin^2 t\right]_0^{2\pi} = -4\pi.$

33. The surface is given by $x + y + z = 1$ or $z = 1 - x - y$, $0 \le x \le 1$, $0 \le y \le 1 - x$ and $\mathbf{r}_x \times \mathbf{r}_y = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Then

$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_D(-y\,\mathbf{i} - z\,\mathbf{j} - x\,\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k})\,dA = \iint_D(-1)\,dA = -(\text{area of } D) = -\frac{1}{2}.$

34. $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E 3(x^2+y^2+z^2)\,dV = \int_0^{2\pi}\int_0^1\int_0^2(3r^2+3z^2)\,r\,dz\,dr\,d\theta = 2\pi\int_0^1(6r^3+8r)\,dr = 11\pi$

35. $\iiint_E \operatorname{div}\mathbf{F}\,dV = \iiint_{x^2+y^2+z^2\le 1} 3\,dV = 3(\text{volume of sphere}) = 4\pi$. Then

$\mathbf{F}(\mathbf{r}(\phi,\theta))\cdot(\mathbf{r}_\phi\times\mathbf{r}_\theta) = \sin^3\phi\,\cos^2\theta + \sin^3\phi\,\sin^2\theta + \sin\phi\,\cos^2\phi = \sin\phi$ and

$\iint_S \mathbf{F}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\pi}\sin\phi\,d\phi\,d\theta = (2\pi)(2) = 4\pi.$

36. Here we must use Equation 17.9.7 [ET 16.9.7] since $\mathbf{F}$ is not defined at the origin. Let S_1 be the sphere of radius 1 with center at the origin and outer unit normal $\mathbf{n}_1$. Let S_2 be the surface of the ellipsoid with outer unit normal $\mathbf{n}_2$ and let E be the solid region between S_1 and S_2. Then the outward flux of $\mathbf{F}$ through the ellipsoid is given by

$\iint_{S_2}\mathbf{F}\cdot\mathbf{n}_2\,dS = -\iint_{S_1}\mathbf{F}\cdot(-\mathbf{n}_1)\,dS + \iiint_E \operatorname{div}\mathbf{F}\,dV$. But $\mathbf{F} = \mathbf{r}/|\mathbf{r}|^3$, so

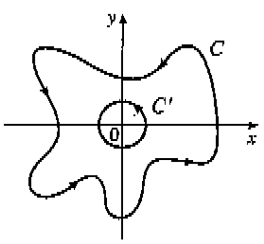
$\operatorname{div}\mathbf{F} = \nabla\cdot\left(|\mathbf{r}|^{-3}\mathbf{r}\right) = |\mathbf{r}|^{-3}(\nabla\cdot\mathbf{r}) + \mathbf{r}\cdot\left(\nabla|\mathbf{r}|^{-3}\right) = |\mathbf{r}|^{-3}(3) + \mathbf{r}\cdot\left(-3|\mathbf{r}|^{-4}\right)\left(\mathbf{r}\,|\mathbf{r}|^{-1}\right) = 0$. [Here we have used Exercises 17.5.30(a) [ET 16.5.30(a)] and 17.5.31(a) [ET 16.5.31(a)].] And $\mathbf{F}\cdot\mathbf{n}_1 = \dfrac{\mathbf{r}}{|\mathbf{r}|^3}\cdot\dfrac{\mathbf{r}}{|\mathbf{r}|} = |\mathbf{r}|^{-2} = 1$ on S_1.

Thus $\iint_{S_2}\mathbf{F}\cdot\mathbf{n}_2\,dS = \iint_{S_1}dS + \iiint_E 0\,dV = (\text{surface area of the unit sphere}) = 4\pi(1)^2 = 4\pi$.

37. Because $\operatorname{curl}\mathbf{F} = \mathbf{0}$, $\mathbf{F}$ is conservative, and if $f(x,y,z) = x^3yz - 3xy + z^2$, then $\nabla f = \mathbf{F}$. Hence

$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \nabla f\cdot d\mathbf{r} = f(0,3,0) - f(0,0,2) = 0 - 4 = -4.$

38. Let C' be the circle with center at the origin and radius a as in the figure. Let D be the region bounded by C and C'. Then D's positively oriented boundary is $C \cup (-C')$. Hence by Green's Theorem



$$\int_C \mathbf{F}\cdot d\mathbf{r} + \int_{-C'}\mathbf{F}\cdot d\mathbf{r} = \iint_D\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA = 0, \text{ so}$$

$\int_C \mathbf{F}\cdot d\mathbf{r} = -\int_{-C'}\mathbf{F}\cdot d\mathbf{r} = \int_{C'}\mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt$

$$= \int_0^{2\pi}\left[\frac{2a^3\cos^3 t + 2a^3\cos t\,\sin^2 t - 2a\sin t}{a^2}(-a\sin t) + \frac{2a^3\sin^3 t + 2a^3\cos^2 t\,\sin t + 2a\cos t}{a^2}(a\cos t)\right]dt$$

$$= \int_0^{2\pi}\frac{2a^2}{a^2}\,dt = 4\pi$$

39. By the Divergence Theorem, $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_E \operatorname{div}\mathbf{F}\,dV = 3(\text{volume of } E) = 3(8-1) = 21$.

40. The stated conditions allow us to use the Divergence Theorem. Hence $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}(\operatorname{curl}\mathbf{F})\,dV = 0$ since $\operatorname{div}(\operatorname{curl}\mathbf{F}) = 0$.

41. Let $\mathbf{F} = \mathbf{a}\times\mathbf{r} = \langle a_1,a_2,a_3\rangle\times\langle x,y,z\rangle = \langle a_2z - a_3y,\, a_3x - a_1z,\, a_1y - a_2x\rangle$. Then $\operatorname{curl}\mathbf{F} = \langle 2a_1, 2a_2, 2a_3\rangle = 2\mathbf{a}$, and $\iint_S 2\mathbf{a}\cdot d\mathbf{S} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C(\mathbf{a}\times\mathbf{r})\cdot d\mathbf{r}$ by Stokes' Theorem.