16B
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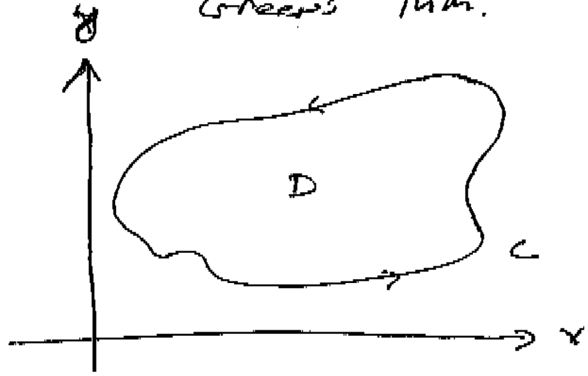
# Stoke's Thm.

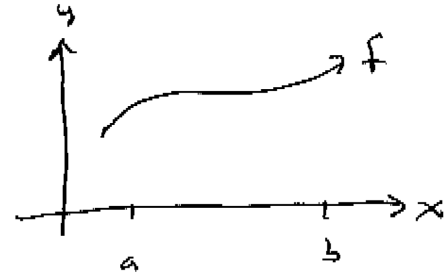
Green's Thm.




derivative $f = F'$

antiderivative on the boundary

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

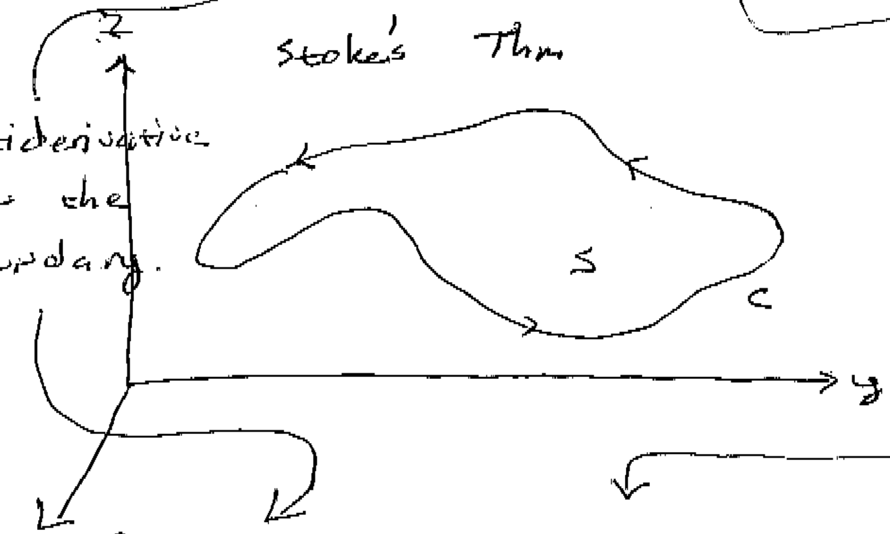



$$\oint_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

Stoke's Thm

antiderivative on the boundary.

"Derivative" inside a region



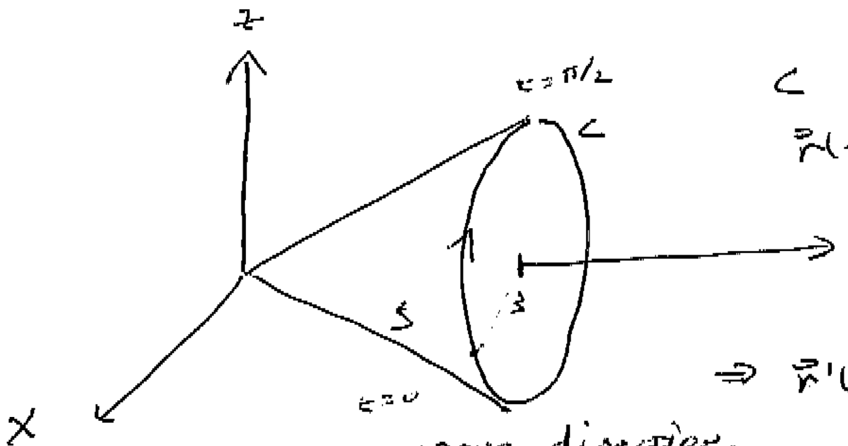

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}(\vec{F})\cdot d\vec{S}$$

<u>Stoke's Thm:</u> Let $S$ be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve $C$ w/positive orientation. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains $S$. Then...

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$$

$\boxed{\begin{array}{c}14.8\\ 2/5\end{array}}$

**Ex 1:** If $\vec{F} = \langle x^2y^3z,\ \sin(xyz),\ xyz \rangle$ & $S$ is the part of a cone $y^2 = x^2 + z^2$ on $0 \le y \le 3$ oriented in the direction of the positive $y$-axis, evaluate $I = \iint_S \text{curl}\,\vec{F} \cdot d\vec{S}$




wrong direction.

$C$ is parametrized by $\vec{r}(t) = \langle 3\cos t,\ 3,\ 3\sin t \rangle$ on $0 \le t \le 2\pi$

$\Rightarrow \vec{r}\,'(t) = \langle -3\sin t,\ 0,\ 3\cos t \rangle$

$$I = \oint_C \vec{F} \cdot d\vec{r} = -\int_0^{2\pi} \vec{F}(\vec{r}(t))\, \vec{r}\,'(t)\, dt$$

$$= -\int_0^{2\pi} \langle 9\cancel{\sin^2}\cos^2(t) \cdot 27 \cdot 3\sin t,\ \sin(27\sin t\cos t),\ 27\sin t\cos t \rangle \cdot \langle -3\sin t, 0, 3\cos t \rangle\, dt$$

$$= -81 \int_0^{2\pi} -27\cos^2 t\sin^2 t + 0 + \sin t\cos^2 t\, dt$$

$$= -81\left[\frac{-\cos^3 t}{3}\right]_0^{2\pi} + 2187\left[\frac{\sin t\cos^3 t}{4} + \frac{1}{4} \cdot \int \cos^2 t\, dt\right]_0^{2\pi}$$

$$= +27(1 - 1) + \frac{2187}{4}\left(\sin t\cos^3 t + \frac{1}{2}t + \frac{1}{4}\sin 2t\right)$$

$$= +\frac{2187}{4}\pi$$

16.8
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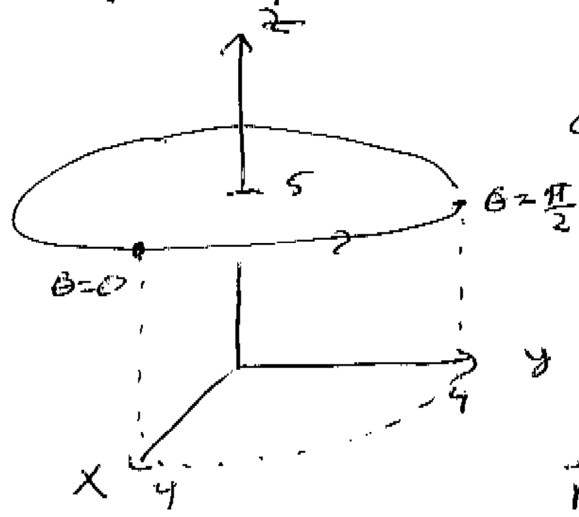
Ex2: If $\vec{F} = \langle yz, 2xz, e^{xy} \rangle$ & C is the circle $x^2 + y^2 = 16$ w/ $z = 5$ traversed CCW. evaluate $\oint_C \vec{F} \cdot d\vec{r}$.




$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 2xz & e^{xy} \end{vmatrix}$$

$$= \langle xe^{xy} - 2x, -ye^{xy} + y, \overbrace{2z - z}^{z} \rangle$$

$$\vec{r}(R, \theta) = \langle R\cos\theta, R\sin\theta, 5 \rangle$$

on $0 \le R \le 4$

$0 \le \theta \le 2\pi$

$$\vec{r}_R = \langle \cos\theta, \sin\theta, 0 \rangle$$

$$\vec{r}_\theta = \langle -R\sin\theta, R\cos\theta, 0 \rangle$$

$$\vec{r}_R \times \vec{r}_\theta = \langle 0, 0, R \rangle$$

$$I = \iint_S \text{curl } \vec{F} \cdot d\vec{S}$$

$$= \iint_D \text{curl } \vec{F} \cdot (\vec{r}_R \times \vec{r}_\theta)\, dA$$

$$= \int_0^4 \int_0^{2\pi} 5R \, d\theta \, dR$$

$$= \int_0^4 10R\pi \, dR$$

$$= 5R^2\pi \Big|_0^4$$

$$= 80\pi$$

$\boxed{\begin{matrix}16.4\\ 4/5\end{matrix}}$

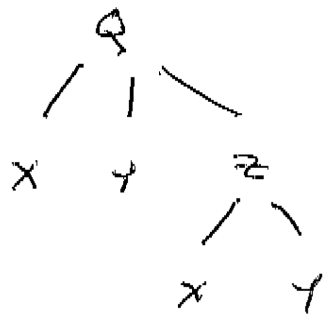
<u>key point</u> w/ the given conditions, Stoke's thm says that $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$ does not depend upon $S$. ~~proo~~ That is

$$\iint_{S_1} \text{curl}\,\vec{F}\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{r} = \iint_{S_2} \text{curl}\,\vec{F}\cdot d\vec{S}$$

if $C$ is the boundary of both $S_1$ & $S_2$.

<u>Recall</u>:

(A) multivariate chain rule



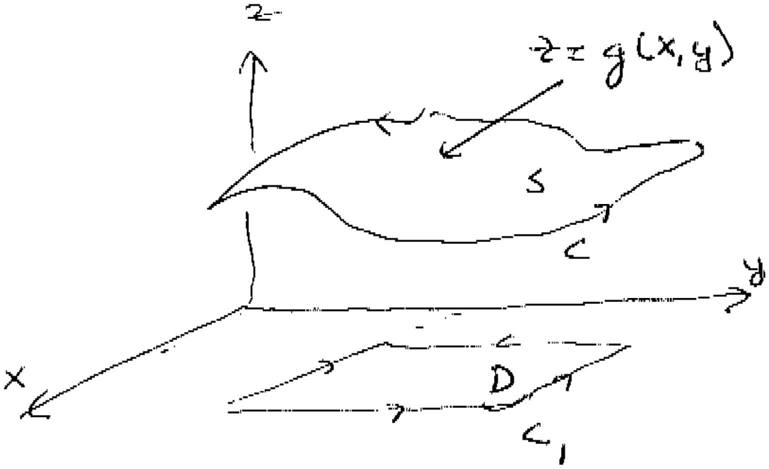

$$\frac{\partial}{\partial x} Q = \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\cdot\frac{\partial z}{\partial y}$$

(B) Green's Thm.

(i) $\oint_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$

(ii) $\oint_C \vec{F}\cdot d\vec{r} = \iint_D \text{curl}\,\vec{F}\cdot\vec{k}\, dA$

If $S$ is a graph & $\vec{F}$, $S$, & $C$ are nice.




$S$ is $z = g(x,y)$ where $(x,y) \in D$.

$C = g(C_1)$ (notation!!)

$C$ & $C_1$ have pos. orientation

$\vec{F} = \langle P, Q, R \rangle$ where the partials are cont.

claim: $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\, \vec{F} \cdot d\vec{S}$

□ Go thru proof in the book. (Tons of references) ■

claim: If $\text{curl}\, \vec{F} = \vec{0}$ on $\mathbb{R}^3$, then $\vec{F}$ is conservative.

□ proof.

Assume $\text{curl}\, \vec{F} = \vec{0}$.

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\, \vec{F} \cdot d\vec{S} \quad \text{(Stoke's Thm)}$$

$$= \iint_S \vec{0} \cdot d\vec{S} \quad \text{(by assumption)}$$

$$= 0$$

so, $\text{curl}\, \vec{F} = \vec{0} \Rightarrow \oint_C \vec{F} \cdot d\vec{r} = 0 \Rightarrow \vec{F}$ is conservative

If $z = g(x, y)$ (p1088)

$\iint_S \vec{F} \cdot d\vec{S} = \iint_D (-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R)\, dA$

If $\vec{F} = \langle P, Q, R \rangle$ (p1062)

Since $S$ is a graph of a function, we can apply Formula 16.7.10 with **F** replaced by curl **F**. The result is

$$\boxed{2} \quad \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

$$\operatorname{curl} \vec{F} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z},\ \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x},\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle = \iint_D \left[ -\left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \frac{\partial z}{\partial x} - \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \frac{\partial z}{\partial y} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \right] dA$$

where the partial derivatives of $P$, $Q$, and $R$ are evaluated at $(x, y, g(x, y))$. If

$$x = x(t) \qquad y = y(t) \qquad a \leqslant t \leqslant b$$

is a parametric representation of $C_1$, then a parametric representation of $C$ is

$$x = x(t) \qquad y = y(t) \qquad z = g(x(t), y(t)) \qquad a \leqslant t \leqslant b$$

This allows us, with the aid of the Chain Rule, to evaluate the line integral as follows:

chain rule

$$\begin{tikzcd} & z \arrow[dl] \arrow[dr] & \\ x \arrow[d] & & y \arrow[d] \\ t & & t \end{tikzcd}$$

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \left( P \frac{dx}{dt} + Q \frac{dy}{dt} + R \frac{dz}{dt} \right) dt \\ &= \int_a^b \left[ P \frac{dx}{dt} + Q \frac{dy}{dt} + R \left( \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \right) \right] dt \\ &= \int_a^b \left[ \left( P + R \frac{\partial z}{\partial x} \right) \frac{dx}{dt} + \left( Q + R \frac{\partial z}{\partial y} \right) \frac{dy}{dt} \right] dt \\ &= \oint_{C_1} \left( P + R \frac{\partial z}{\partial x} \right) dx + \left( Q + R \frac{\partial z}{\partial y} \right) dy \\ &= \iint_D \left[ \frac{\partial}{\partial x} \left( Q + R \frac{\partial z}{\partial y} \right) - \frac{\partial}{\partial y} \left( P + R \frac{\partial z}{\partial x} \right) \right] dA \end{aligned}$$

Green's Thm. (p1055)

$\oint_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$

Chain Rule (p904)

$$\begin{tikzcd} & Q \arrow[dl] \arrow[d] \arrow[dr] & & \\ x & y & z \arrow[dl] \arrow[dr] & \\ & x & & y \end{tikzcd}$$

$\frac{\partial}{\partial x} Q = \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z} \frac{\partial z}{\partial x}$

where we have used Green's Theorem in the last step. Then, using the Chain Rule again and remembering that $P$, $Q$, and $R$ are functions of $x$, $y$, and $z$ and that $z$ is itself a function of $x$ and $y$, we get

product rule

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left[ \left( \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial R}{\partial x} \frac{\partial z}{\partial y} + \frac{\partial R}{\partial z} \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} + R \frac{\partial^2 z}{\partial x\, \partial y} \right) - \left( \frac{\partial P}{\partial y} + \frac{\partial P}{\partial z} \frac{\partial z}{\partial y} + \frac{\partial R}{\partial y} \frac{\partial z}{\partial x} + \frac{\partial R}{\partial z} \frac{\partial z}{\partial y} \frac{\partial z}{\partial x} + R \frac{\partial^2 z}{\partial y\, \partial x} \right) \right] dA$$

Four of the terms in this double integral cancel and the remaining six terms can be arranged to coincide with the right side of Equation 2. Therefore

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} \qquad \square$$