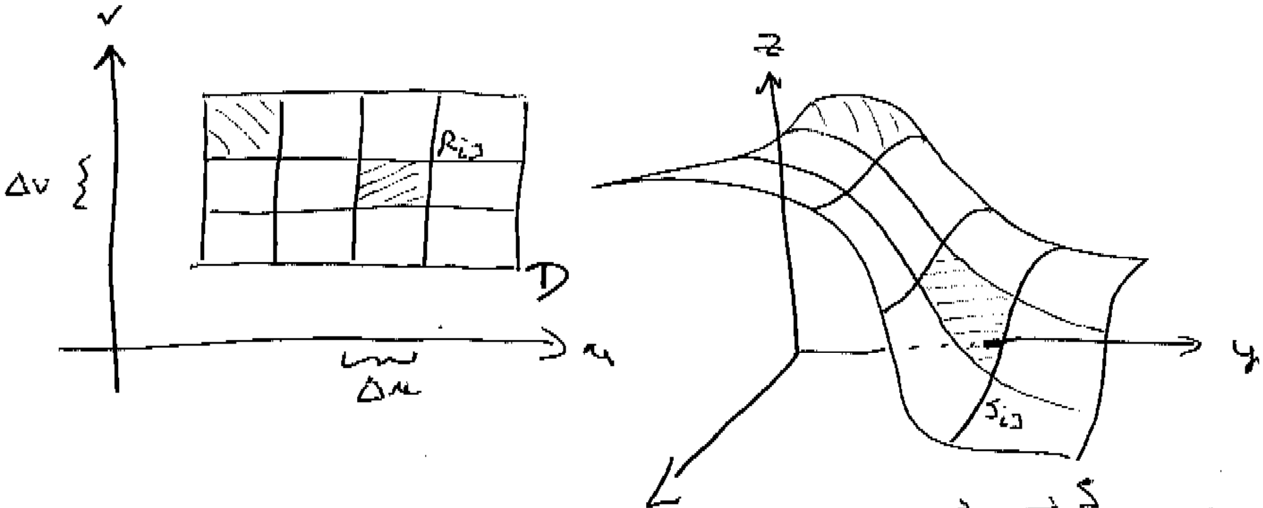
16.7: Surface Integrals

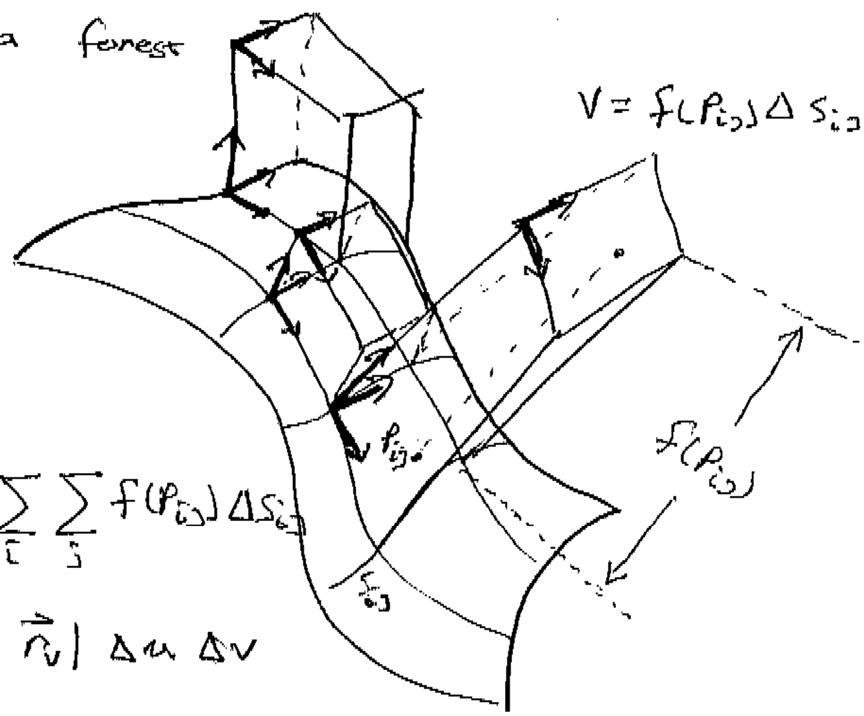
(I) parametric Surfaces

$$\vec{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v) \rangle$$



$$\Delta S_{ij} \approx |\vec{r}_u \times \vec{r}_v| \,\Delta u \,\Delta v$$

Instead of the area of the Great wall... think the volume of a forest on a hill side.



points (x,y,z)

$$V \approx \sum_i \sum_j f(P_{ij}) \Delta S_{ij}$$

AND

$$\iint_S f(x,y,z)\,dS = \lim \sum_i \sum_j f(P_{ij}) \Delta S_{ij}$$

recall $\Delta S_{ij} = |\vec{r}_u \times \vec{r}_v| \,\Delta u\, \Delta v$

so $$\iint_S f(x,y,z)\,dS = \iint_D f(\cancel{x,y,z})^{\vec{r}(u,v)} |\vec{r}_u \times \vec{r}_v| \,dA$$

WRT surface area — WRT the parameters

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Ex1: $\iint_S yz\,dS$, S is the surface w/ parametric equations $\vec{r}(u,v) = \langle u^2, u\sin v, u\cos v\rangle$ on $0 \le u \le 1$, $0 \le v \le \pi/2$.

$$r(u,v) = \langle u^2, u\sin v, u\cos v\rangle \qquad 0 \le u \le 1$$

$$0 \le v \le \frac{\pi}{2}$$

$$\vec{r}_u = \langle 2u, \sin v, \cos v\rangle$$

$$\vec{r}_v = \langle 0, u\cos v, -u\sin v\rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2u & \sin v & \cos v \\ 0 & u\cos v & -u\sin v \end{vmatrix}$$

$$= \langle \underbrace{-u\sin^2 v - u\cos^2 v}_{-u.}, 2u^2\sin v, 2u^2\cos v\rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{u^2 + 4u^4\sin^2 v + 4u^4\cos^2 v}$$

$$= u\sqrt{1 + 4u^2}$$

$$f(x,y,z) = yz$$

AND $f(\vec{r}(u,v)) = u\sin v \cdot u\cos v$

$$= u^2 \sin v \cos v$$

SO $$I = \int_0^{\frac{\pi}{2}}\int_0^1 u^3 \sin v\cos v\sqrt{1+4u^2}\,du\,dv$$

$$= \int_0^{\frac{\pi}{2}} \sin v\cos v\,dv \cdot \int_0^1 u^3\sqrt{1+4u^2}\,du.$$

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$$= \left[\frac{\sin^2 v}{2}\right]_0^{\pi/2} \cdot \int_1^5 \frac{1}{8}\cdot\left(\frac{w-1}{4}\right)\sqrt{w}\,dw$$

$$= \frac{1}{2} \cdot \frac{1}{32}\int_1^5 w^{3/2} - w^{1/2}\,dw$$

$$= \frac{1}{2} \cdot \frac{1}{32}\left[\frac{2}{5}w^{5/2} - \frac{2}{3}w^{3/2}\right]_1^5$$

$$= \frac{1}{64}\left[\frac{2}{5}\cdot 5^{5/2} - \frac{2}{3}\cdot 5^{3/2} - \frac{2}{5} + \frac{2}{3}\right]$$

$$= \frac{5\sqrt{5}}{48} + \frac{1}{240}$$

Derivation when S is the graph of a fct $x = g(y,z)$. Parametrize ...

$$\vec{r}(y,z) = \langle g(y,z), y, z\rangle$$

$$\vec{r}_y = \langle g_y(y,z), 1, 0\rangle$$

$$\vec{r}_z = \langle g_z(y,z), 0, 1\rangle$$

$$\vec{r}_y \times \vec{r}_z = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial g}{\partial y} & 1 & 0 \\ \frac{\partial g}{\partial z} & 0 & 1 \end{vmatrix} = \left\langle 1, -\frac{\partial g}{\partial y}, -\frac{\partial g}{\partial z}\right\rangle$$

w/ norm: $\sqrt{1 + (g_y)^2 + (g_z)^2}$

AND $\iint_S f(x,y,z)\,dS = \iint_D f(g(y,z),y,z)\sqrt{1+(g_y)^2+(g_z)^2}\,dA$

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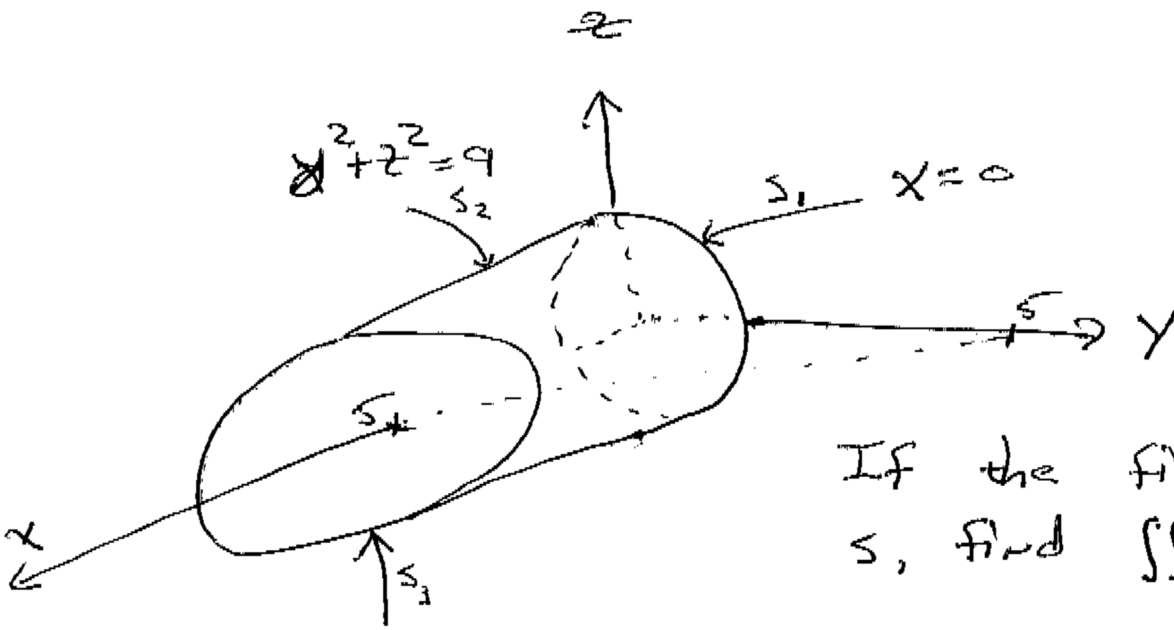

If the figure represents S, find $\iint_S xz\,dS$

$x = 5 - y$.

$$I = \iint_S xz\,ds = \underbrace{\iint_{S_1} xz\,ds}_{0} + \iint_{S_2} xz\,ds + \iint_{S_3} xz\,ds$$

$$S_2: \quad \vec{r}(u,\theta) = \langle u,\ 3\cos\theta,\ 3\sin\theta \rangle$$

$$\vec{r}_u = \langle 1, 0, 0 \rangle$$

$$\vec{r}_\theta = \langle 0, -3\sin\theta, 3\cos\theta \rangle$$

$$\vec{r}_u \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & -3\sin\theta & 3\cos\theta \end{vmatrix}.$$

relates to the position vector.

$$= \langle 0, -3\cos\theta, -3\sin\theta \rangle$$

$$|\vec{r}_u \times \vec{r}_\theta| = 3.$$

$$I_2 = \iint_{S_2} xz\,ds = \int_0^{2\pi}\int_0^{5-3\cos\theta} u \cdot 3\sin\theta \cdot \cancel{3}\,\cancel{dA}\ du\,d\theta$$

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$$\hookrightarrow \quad I_2 = 9 \int_0^{2\pi} \left[\frac{u^2}{2} \right]_0^{5-3\cos\theta} \sin\theta \; d\theta$$

$$= 9 \int_0^{2\pi} \frac{(5-3\cos\theta)^2}{2} \sin\theta \; d\theta$$

Let $w = \cos\theta \Rightarrow -dw = \sin\theta \, d\theta$

$$= -9 \int_1^1 \frac{(5-3w)^2}{2} dw$$

$$= 0$$

S_3 : $x = 5 - y$ or $r(y,z) = \langle 5-y, y, z \rangle$

$$I_3 = \iint_{S_3} xz \, dS = \iint_D (5-y) \cdot z \sqrt{1 + 1^2 + 0^2} \, dA$$

D : $y^2 + z^2 \le 9$

$$= \sqrt{2} \iint_D 5z - zy \; dA$$

$y = r\cos\theta \qquad z = r\sin\theta$

$$= \sqrt{2} \int_0^{2\pi} \int_0^3 (5r\sin\theta - r^2 \sin\theta\cos\theta) \cdot r \, dr \, d\theta$$

$$= \sqrt{2} \int_0^{2\pi} \left[\frac{5r^3}{3} \sin\theta - \frac{r^4}{4} \sin\theta\cos\theta \right]_0^3 d\theta$$

$$= \sqrt{2} \int_0^{2\pi} \left(45 \sin\theta - \frac{81}{4} \sin\theta\cos\theta \right) d\theta$$

$$= 0.$$

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Oriented surfaces

Mobius strip.

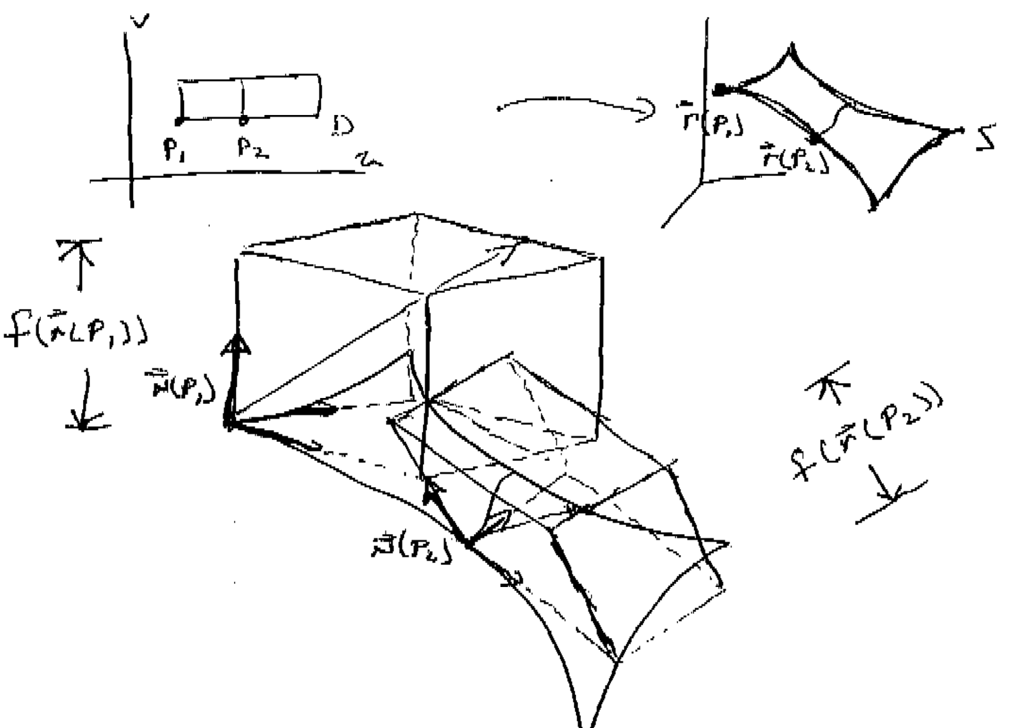
which way is up?

Graph .... closed surface.

$$\vec{N} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \quad (\text{surface of } \vec{r}(u,v))$$

$$\vec{N} = \frac{\langle 1, -g_y, -g_z \rangle}{\sqrt{1+(g_y)^2+(g_z)^2}} \quad \text{if } x = g(y,z) \text{ (a graph)}$$

Surface Integrals of vector fields



If $\vec{N}$ is a unit vector, $f(r(P)) = \vec{F}(\vec{r}(P)) \cdot \vec{N}(P)$

or $\vec{F} \cdot \vec{N}$

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Defn: If $\vec{F}$ is a cont. vector field defined on an oriented surface S w/ unit normal vector $\vec{n}$, then the surface integral of $\vec{F}$ over S is

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} \, dS$$

this is the flux of $\vec{F}$ across S.

Fish net analogy: The flux gives the rate of flow of a liquid w/ density 1 in units of mass/time.

to simplify calculations... If S is given by $\vec{r}(u,v)$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \, dS$$

$$= \iint_D \vec{F}(\vec{r}(u,v)) \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \, |\vec{r}_u \times \vec{r}_v| \, dA$$

"volume" of a parallel piped.

$$= \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) \, dA$$

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$$= 625 \int_0^{\pi} \left(-\frac{\pi}{2} \sin^3\phi \cos\phi + \frac{-4}{\cancel{20}} \sin^2\phi \cos\phi \right) d\phi$$

Let $u = \sin\phi$
$du = \cos\phi \, d\phi$

$$= 625 \int_0^0 -\frac{\pi}{2} u^3 \frac{-4}{\cancel{+20}} u^2 \, du$$

$$= 0$$

AND if $\vec{F} = \langle P, Q, R \rangle$ & $x = g(y,z)$

$$\vec{F} \cdot (\vec{r}_y \times \vec{r}_z) = \langle P, Q, R \rangle \cdot \langle 1, -g_y, -g_z \rangle$$

$$= P - Q g_y - R g_z$$

so $$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \left(P - Q \frac{\partial g}{\partial y} - R \frac{\partial g}{\partial z} \right) dA$$

Ex 4: set up an integral to evaluate

$I = \iint_S xyz \, dS$ where $z = xy$ on $0 \le x, y \le 1$.

$r(x,y) = \langle x, y, xy \rangle$

$$I = \int_0^1 \int_0^1 xy \overbrace{(xy)}^{z} \sqrt{1 + y^2 + x^2} \, dx \, dy.$$

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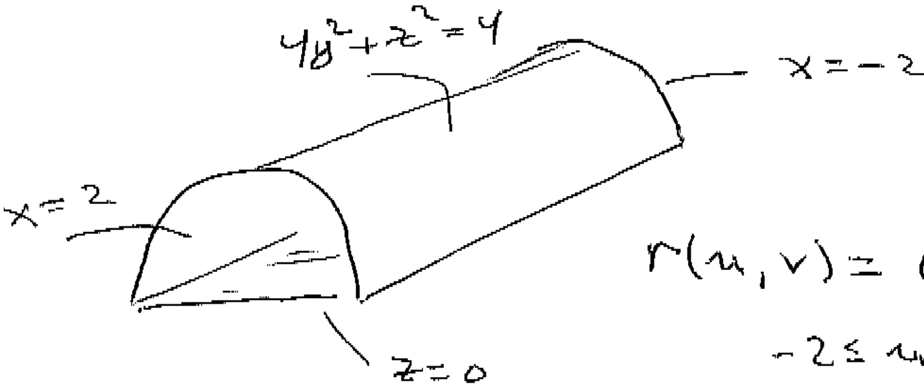
Ex 5: Set up an integral to find the flux of $\vec{F} = \langle \sin(xyz), x^2 y, z^2 e^{x/5} \rangle$ across $4y^2 + z^2 = 4$ where $z \geq 0$ and $-2 \leq x \leq 2$.



$r(u,v) = \langle u, \cos v, 2\sin v \rangle$

$-2 \leq u \leq 2$ & $0 \leq v \leq \pi$

$I = \iint_S \vec{F} \cdot n \, dS$.

use a different parametrization. $\boxed{\text{Something is wrong}}$

$F(r(u,v)) = \langle \sin(2u \sin v \cos v), u^2 \cos v, 4\sin^2 v \, e^{u/5} \rangle$

$\vec{r}_u = \langle 1, 0, 0 \rangle$

$\vec{r}_v = \langle 0, -\sin v, 2\cos v \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 0, -2\cos v, -\sin v \rangle$

w/ magnitude $\sqrt{4\cos^2 v + \sin^2 v}$ (not necessary)

$I = \int_0^{\pi} \int_0^1 -2u^2\cos^2 v - 4\sin^3 v \, e^{u/5} \, du \, dv.$

$\underbrace{\vec{F}(r(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)}$

$$I = \int_0^{\pi} \left[-\frac{2}{3} u^3 \cos^2 v - 20 \sin^3 v \, e^{u/5} \right]_0^1 dv$$

$$= \int_0^{\pi} -\frac{2}{3} \cos^2 v - 20 \sin^3 v \, e^{1/5} + 20 \sin^3 v \; dv$$

$$= 2 \int_0^{\pi} \frac{1}{3} \cos^2 v + \cancel{10} \, 10 \sin^3 v \left(1 - e^{1/5}\right) dv$$

$$= 2 \left[\frac{\frac{1}{2} v + \frac{1}{4} \sin(2v)}{3} + 10\left(1 - e^{1/5}\right)\left(-\frac{1}{3}(2 + \sin^2 v) \cdot \cos(v)\right) \right]_0^{\pi}$$

$$= 2\left(\frac{\frac{1}{2}(\pi) + \frac{1}{4} \sin(2\pi)}{3} + 10\left(1 - e^{1/5}\right)\left(-\frac{1}{3}\left(2 + \sin^2(\pi)\right) \cdot \cos(\pi)\right)\right)$$

$$- 2\left(\frac{0 + 0}{3} + 10\left(1 - e^{1/5}\right)\left(-\frac{1}{3}(2) \cos(0)\right)\right)$$

$$= \frac{\pi}{3} + \frac{20}{3}\left(1 - e^{1/5}\right) + \frac{20}{3}\left(1 - e^{1/5}\right)$$