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16.6: Parametric Surfaces & ~~Their~~ Their Areas.

To date: $r(u) = \langle x(u), y(u), z(u) \rangle$

Now $r(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

Ex1: Basic Fcts. $z = f(x,y)$.

Ex2: Great wall: $r(u,v) = \langle x(u), y(u), v\, f(x(u), y(u$

Ex3: Surfaces of revolution.

Ex4: Line in $\mathbb{R}^3$

Find parametric Eqts

Ex5: The part of $y^2 + z^2 = 16$
where $0 \le x \le 5$

Ex6: $x^2 + y^2 - z^2 = 1$ & $y \ge 0$

if $z = t$

$x = \sqrt{1+t^2}\cos(\theta)$

$y = \sqrt{1+t^2}\sin(\theta)$, $0 \le \theta \le \pi$

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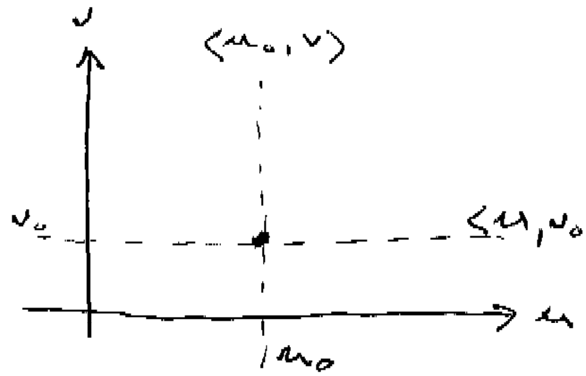
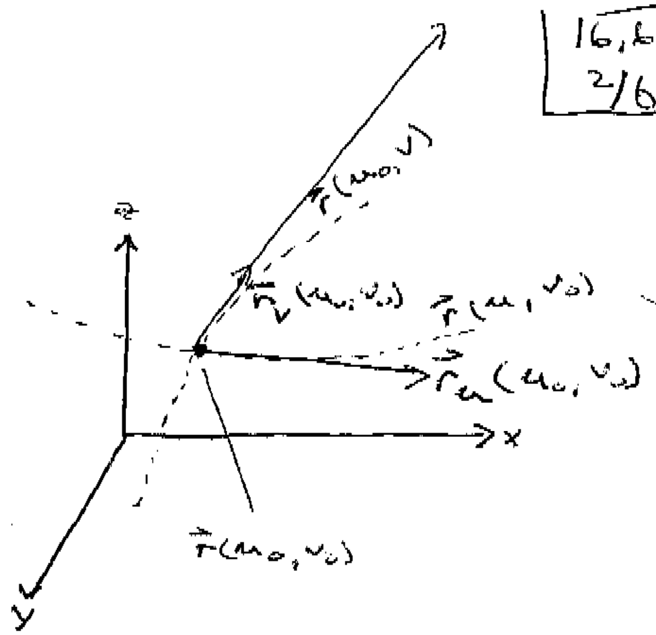
Tangent Planes





$$\vec{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$$

$$\vec{r}_u(u_0,v_0) = \langle x_u(u_0,v_0),\ y_u(u_0,v_0),\ z_u(u_0,v_0)\rangle$$

$$\vec{r}_v(u_0,v_0) = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right\rangle \Bigg|_{(u,v)=(u_0,v_0)}$$

Ex: Find an eqt of the tangent plane to $\vec{r}(u,v) = \langle uv,\ u\sin v,\ v\,\cancel{\sin}\overset{\cos}{}\, u\rangle$ at $(0,\pi)$

(B)
$$\vec{r}_u(u,v) = \langle v,\ \sin v,\ \cancel{v\cos u}\ -v\sin u\rangle$$
$$\&\ \vec{r}_u(0,\pi) = \langle \pi,\ 0,\ \cancel{\pi}\ 0\rangle$$
$$\vec{r}_v(u,v) = \langle u,\ u\cos v,\ \cos u\rangle$$
$$\&\ \vec{r}_v(0,\pi) = \langle 0,\ 0,\ 1\rangle$$

vectors on the plane

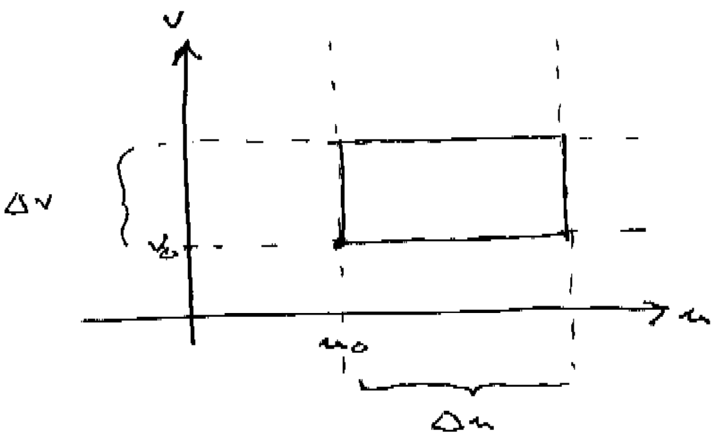
(A) $\vec{r}(0,\pi) = \langle 0, 0, \pi\rangle$ ⟵ pt on the plane

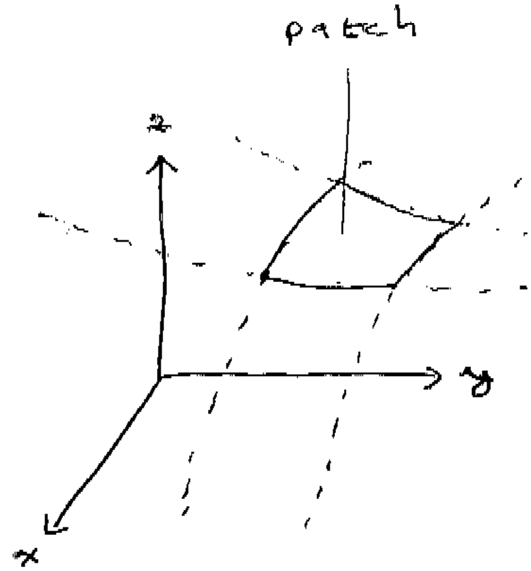
(C) $\langle \pi,0,0\rangle \times \langle 0,0,1\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \pi & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\pi\vec{j}$ is normal to the plane.

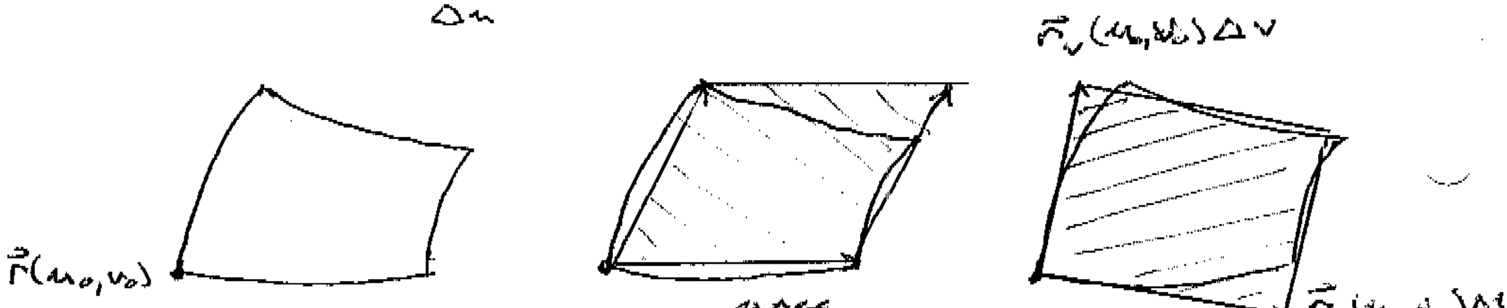
(D) $0(x-0) - \pi(y-0) + 0(z-\pi) = 0$ (plane)

A more interesting example is finding the tangent plane when $(u, v) = (1, 1)$ to $\vec{r}(u, v) = \langle u^2, v^2, uv \rangle$

Surface Area







$A =$ exact area of the patch

$A \approx$ ~~approx~~ area of the "secant patch"

$A \approx$ area of the "tangent patch"

so, the area of our patch is

$$A \approx |\vec{r}_u(u_0, v_0)\Delta u \times \vec{r}_v(u_0, v_0)\Delta v|$$
$$= |\vec{r}_u(u_0, v_0) \times \vec{r}_v(u_0, v_0)|\, \Delta u\, \Delta v.$$

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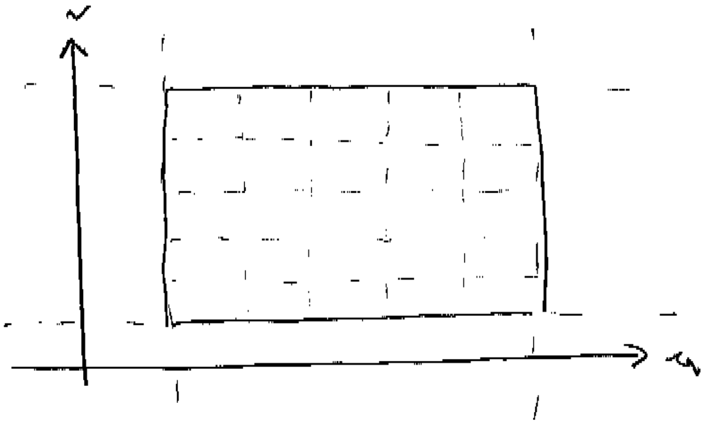
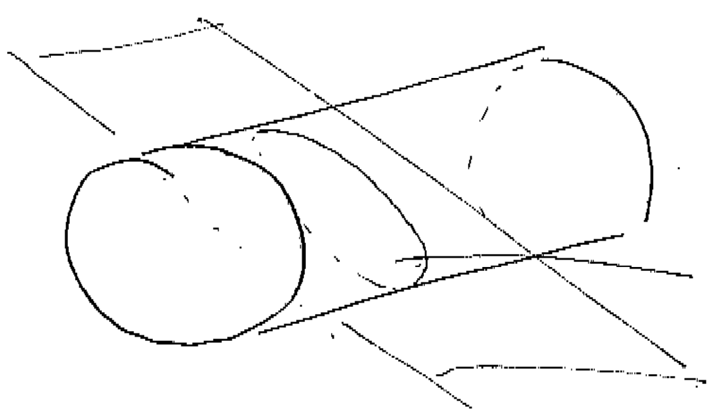


$$A \approx \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} |\vec{r}_u(u_i, v_j) \times \vec{r}_v(u_i, v_j)| \, \Delta u \Delta v.$$

AND if $(\Delta u, \Delta v) \longmapsto (0,0)$

$$A = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA$$

Ex : Find the area of ~~the~~ $2x + 5y + z = 10$ inside $x^2 + y^2 = 9$



$z = 10 - 2x - 5y$

$u = x$

$v = y$

area of an ellipse.

$r(u,v) = \langle u, v, 10 - 2u - 5v \rangle$

$\vec{r}_u = \langle 1, 0, -2 \rangle$ & $\vec{r}_v = \langle 0, 1, -5 \rangle$

$$|\vec{r}_u \times \vec{r}_v| = \left\| \begin{vmatrix} i & j & k \\ 1 & 0 & -2 \\ 0 & 1 & -5 \end{vmatrix} \right\| = |\langle 2, 5, 1 \rangle| = \sqrt{30}$$

$$A = \iint_D \sqrt{30} \, dA = \sqrt{30} \cdot \pi \cdot 3^2 = 9\sqrt{30}\,\pi$$

Notice that this is an example of finding the SA of the graph of a fct $z = f(x,y)$.

$$\vec{r}(x,y) = \langle x, y, f(x,y) \rangle$$

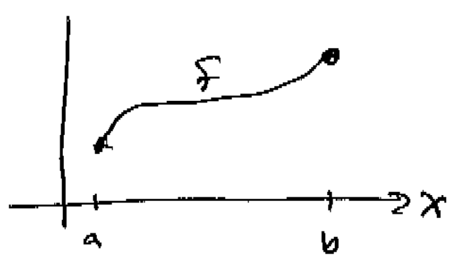
AND
$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix}$$

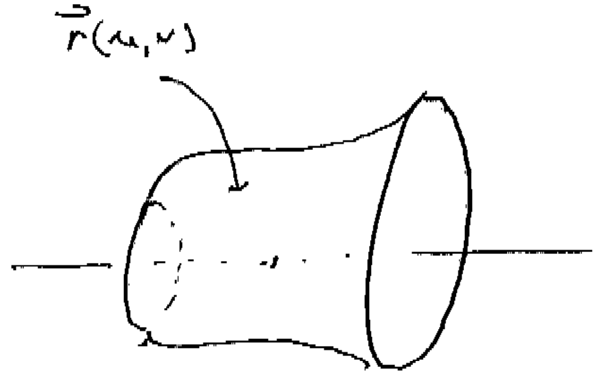
$$= \langle f_x, f_y, 1 \rangle$$

$$\Rightarrow \quad |\vec{r}_x \times \vec{r}_y| = \sqrt{1 + (f_x)^2 + (f_y)^2}$$

so
$$A = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dA.$$

Finally, what is SA of the surface formed by rotating $y = f(x)$ about the x-axis on $a \le x \le b$.





$$\vec{r}(u,\theta) = \langle u, f(u)\cos\theta, f(u)\sin\theta \rangle$$

where $a \le u \le b$ & $0 \le \theta \le 2\pi$

$$r_u = \langle 1, f_u \cos\theta, f_u \sin\theta \rangle$$

$$r_\theta = \langle 0, -f\sin\theta, f\cos\theta \rangle$$

$$\vec{r}_u \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & f_u\cos\theta & f_u\sin\theta \\ 0 & -f\sin\theta & f\cos\theta \end{vmatrix}$$

$$= \langle f \cdot f_u \cos^2\theta + f\cdot f_u \sin^2\theta, \; -f\cos\theta, -f\sin\theta \rangle$$

$$= \langle f\cdot f_u, -f\cos\theta, f\sin\theta \rangle$$

$$= f\langle f_u, -\cos\theta, \sin\theta \rangle$$

AND $$|\vec{r}_u \times \vec{r}_\theta| = f\sqrt{\left(\frac{\partial f}{\partial u}\right)^2 + \cos^2\theta + \sin^2\theta}$$

$$= f(x)\sqrt{1 + \left(\frac{df}{dx}\right)^2}$$

so $$SA = \iint_D f(x)\sqrt{1+\left(\frac{df}{dx}\right)^2}\, dA$$

$$= \int_a^b \int_0^{2\pi} f(x)\sqrt{1+(f'(x))^2}\, d\theta\, dx$$

$$= 2\pi \int_a^b f(x)\sqrt{1+[f'(x)]^2}\, dx$$