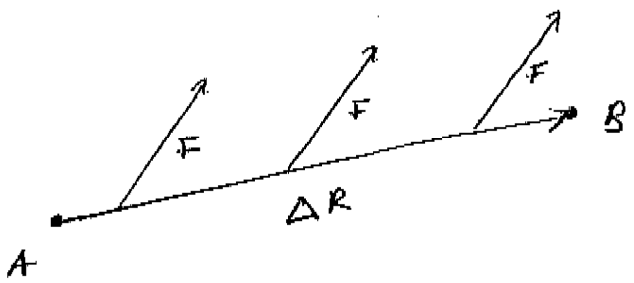
16.2: Line Integrals

From the perspective of work

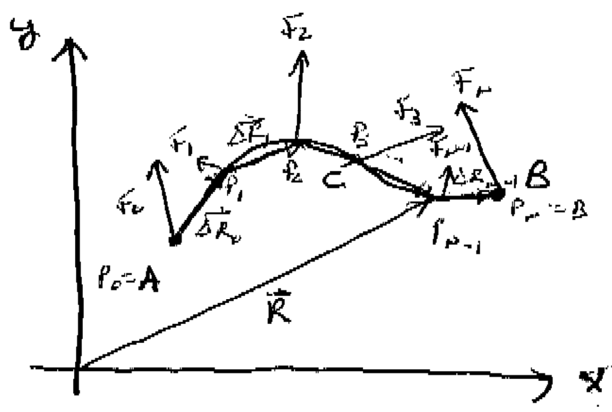


The work to move a particle from A to B by the constant force F is $W = F \cdot \Delta R$.

Now suppose that F is not constant and that the particle does not move along a straight line.

$$\vec{F} = \vec{F}(x,y) = M(x,y)\vec{i} + N(x,y)\vec{j}$$

and the particle moves along C where C is parametrized by $(x(t), y(t))$ on $t_1 \le t \le t_2$.



$$W \approx \sum_{k=0}^{n-1} \vec{F}_k \cdot \overrightarrow{\Delta R_k}$$

AND

$$\int_C \vec{F} \cdot \overrightarrow{dR} = \lim_{n \to \infty} \sum_{k=0}^{n-1} \vec{F}_k \cdot \overrightarrow{\Delta R_k}$$

so $dW = \vec{F} \cdot d\vec{R}$ AND $W = \int dW = \int_C \vec{F} \cdot d\vec{R}$

we think of the position vector $\vec{R}$ as a fct of the arclength s measured from the initial point A. Since we know $\frac{d\vec{R}}{ds}$ is the unit tangent vector (sec 13.2 and 13.3) $\vec{T}$, we can write

$$* \quad \int_C \vec{F} \cdot d\vec{R} = \int_C \vec{F} \cdot \frac{d\vec{R}}{ds}\, ds = \int_C \vec{F} \cdot \vec{T}\, ds$$

"The line integral can be thought of as the integral of the tangential component $\vec{F}$ along the curve C."

Special case: If C is along the x axis from a to b & $\vec{F} = f(x)\,i$ (force in the direction of travel) $W = \int_a^b f(x)\,dx$

Problem: This previous formula $*$ is difficult to work w/.

recall: $\vec{F}(x,y) = M(x,y)\vec{i} + N(x,y)\vec{j}$

$$\vec{R} = x\vec{i} + y\vec{j}$$

AND $d\vec{R} = dx\,\vec{i} + dy\,\vec{j}$

AND C is parametrized by $(x(t), y(t))$ on $t_1 \le t \le t_2$

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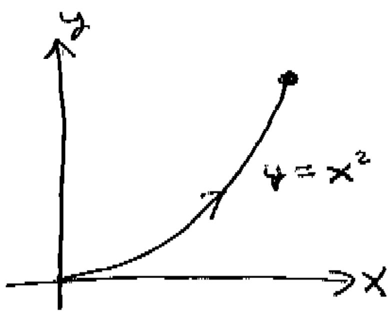
so $\vec{F} \cdot d\vec{R} = M(x,y)\,dx + N(x,y)\,dy$

$$\Rightarrow \int_C \vec{F} \cdot d\vec{R} = \int_C \underbrace{M(x,y)\,dx}_{M(x,y)\frac{dx}{dt}\cdot dt} + \underbrace{N(x,y)\,dy}_{N(x,y)\frac{dy}{dt}\,dt}$$

$$= \int_{t_1}^{t_2} \left[M(x,y)\frac{dx}{dt} + N(x,y)\frac{dy}{dt} \right] dt$$

This is a single integral w/ one variable t.

<u>Ex1</u>: Evaluate $I = \int_C x^2 y\,dx + (x-y)\,dy$ where C is the segment of $y = x^2$ from $(0,0)$ to $(1,1)$.



parametrization

$x = t$ & $y = t^2$ on $0 \le t \le 1$

$dx = dt \qquad dy = 2t\,dt$

$$I = \int_0^1 t^2 \cdot t^2\,dt + (t - t^2)\,dt$$
$$= -2/15$$

<u>Ex1 rev</u>: Same integral ... different parametrization

$x = \sin t$ & $y = \sin^2 t$ on $0 \le t \le \frac{\pi}{2}$

<u>PT</u>: The parametrization doesn't matter so long as the direction stays the same

<u>NOTE</u>: $\int_{-C} \vec{F} \cdot d\vec{R} = -\int_C \vec{F} \cdot d\vec{R}$

Ex 2: Evaluate $I = \int_C x^2 y\, dx + (x-2y)\, dy$ where C is the straight line segment from $(0,0)$ to $(1,1)$.

This is the same integrand as in ex 1.

$x = t$ & $y = t$ on $0 \le t \le 1$

$\Rightarrow dx = dt$ & $dy = dt$

AND $I = \int_0^1 t^2 \cdot t\, dt + (t-2t)\, dt = -\frac{1}{4}$

PT: different paths may lead to different values.

NOTE: In the previous two examples we would say $m(x,y) = x^2 y$ & $n(x,y) = x - 2y$ OR we might say the vector field

$\vec{F} = x^2 y \vec{i} + (x-2y)\vec{j}$

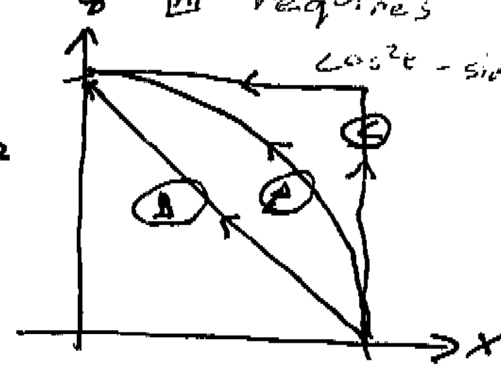
Ex 3: Evaluate $I = \int_C y\, dx + (x+2y)\, dy$ over three curves

[A] C_A: $(\cos t, \sin t)$ on $0 \le t \le \pi/2$

C_B: $(t, 1-t)$ on $0 \le t \le 1$

C_C: $\begin{cases} (1, t), & 0 \le t \le 1 \\ (2-t, 1), & 1 \le t \le 2 \end{cases}$

[A] requires $\cos^2 t - \sin^2 t = \cos 2t$



Ans: 1 in all 3 cases.

PT: All three integrals have the same value & have the same start/end points ... would any curve starting @ (1,0) & ending @ (0,1) have the same value? (see ex 2) ... Yes - for this integral.

This is a foreshadow of cool stuff to come regarding conservative vector fields.

Finally, lets talk about closed curves (same start and end). These are noted as $\oint$.

Ex 4: Calculate $\oint \vec{F} \cdot d\vec{R}$ where $\vec{F} = y\vec{i} + 2x\vec{j}$ where C is the unit circle traversed c.c.w

$x = \cos t$ & $y = \sin t$ on $0 \le t \le 2\pi$

$\Rightarrow dx = -\sin t\, dt$ & $dy = \cos t\, dt$

requires half angle formulas Ans $= \pi$

$\cos^2\theta = \dfrac{1 + \cos 2\theta}{2}$ and $\sin^2\theta = \dfrac{1 - \cos 2\theta}{2}$