16.1: Vector Fields

Velocity vector fields (wind patterns)
(ocean currents)

Force fields (gravitation)
(magnetic)

Defn. Let D be a set in $\mathbb{R}^2$. A vector field on $\mathbb{R}^2$ is a fct $\vec{F}$ that assigns $(x,y)\in D$ a vector $\vec{F}(x,y)$

$$\vec{F}: (x,y) \longmapsto \langle P(x,y), Q(x,y)\rangle$$

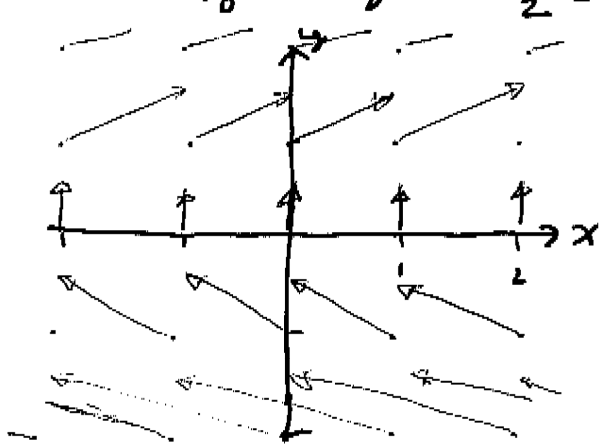
or $\vec{F} = P\vec{i} + Q\vec{j}$

Similarly, if $E \subset \mathbb{R}^3$ ...

$$\vec{F}(x,y,z) \longmapsto \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$$

or $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$

Ex1: $\vec{F}(x,y) = y\vec{i} + \frac{1}{2}\vec{j}$



Scaled v. Unscaled vector Fields

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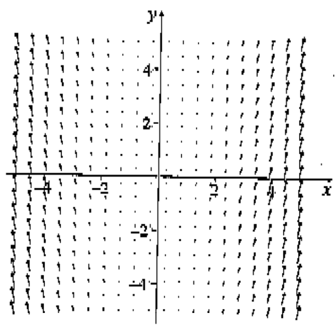
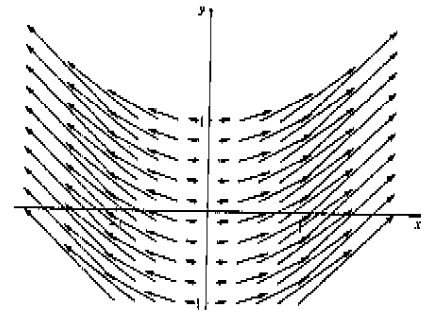
1. $\mathbf{F}(x, y) = x\,\mathbf{i} + x^2\,\mathbf{j}$

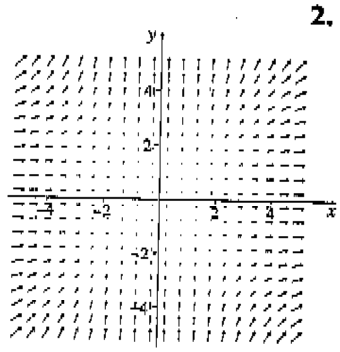


Scaled

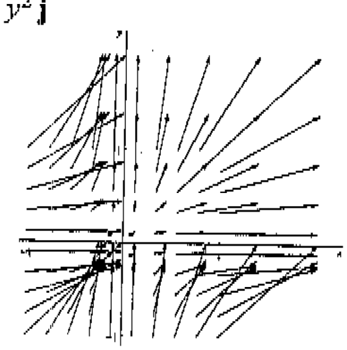


Unscaled

2. $\mathbf{F}(x, y) = x^2\,\mathbf{i} + y^2\,\mathbf{j}$

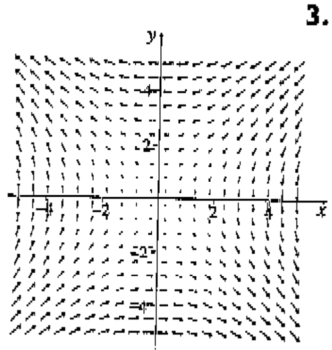


Scaled

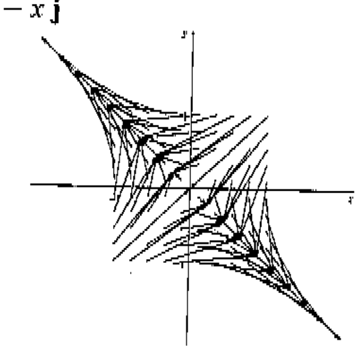


Unscaled

3. $\mathbf{F}(x, y) = -y\,\mathbf{i} - x\,\mathbf{j}$



Scaled



Unscaled

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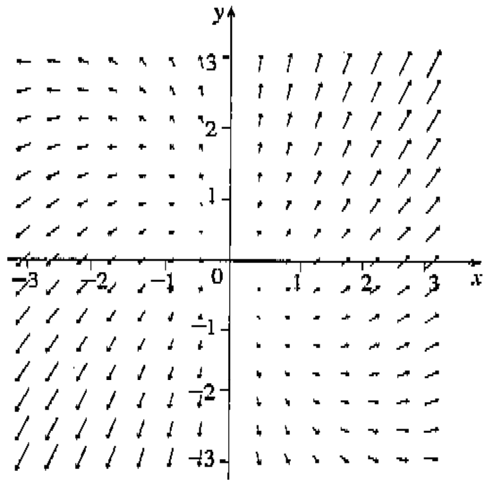
MATCH

(A) $\langle x, x-y \rangle$

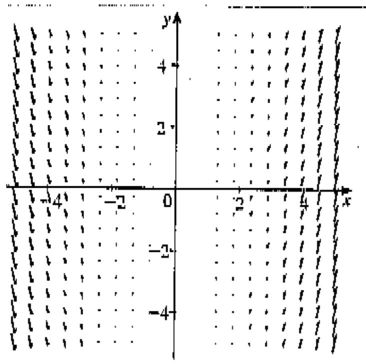
(B) $\langle y, x-y \rangle$

(C) $\langle x, x+y \rangle$

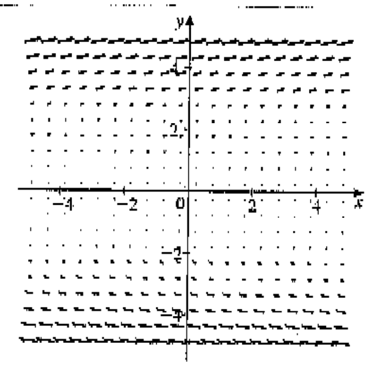
(D) $\langle y, x+y \rangle$



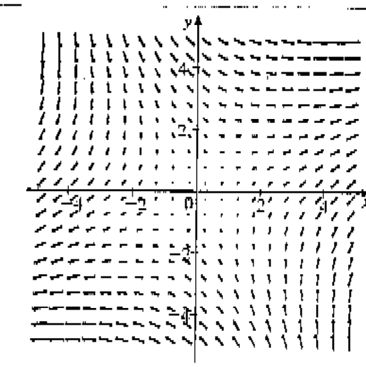
Interesting Vector Fields.



$\mathbf{F}(x, y) = x^2\,\mathbf{i} + x^3\,\mathbf{j}$



$\mathbf{F}(x, y) = y^3\,\mathbf{i} + y^2\,\mathbf{j}$



$\mathbf{F}(x, y) = (x + y)\,\mathbf{i} + (x - y)\,\mathbf{j}$
(Plot along the line $y = mx$ for various values of m.)

$\boxed{\begin{matrix}16.1 \\ 4/6\end{matrix}}$

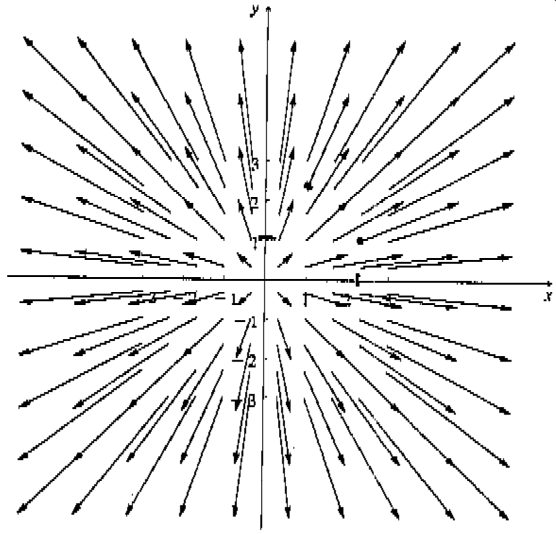
MATCH

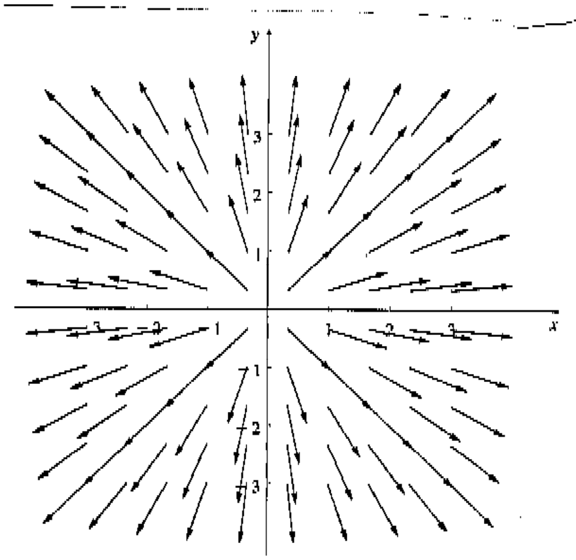
(A) $\langle x, y \rangle$

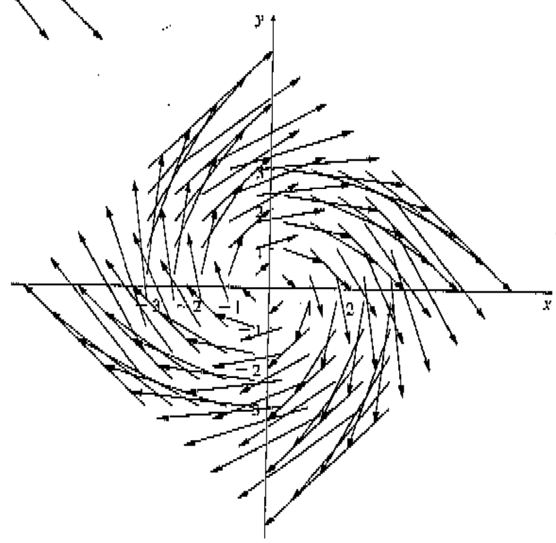
(B) $\langle y, -x \rangle$

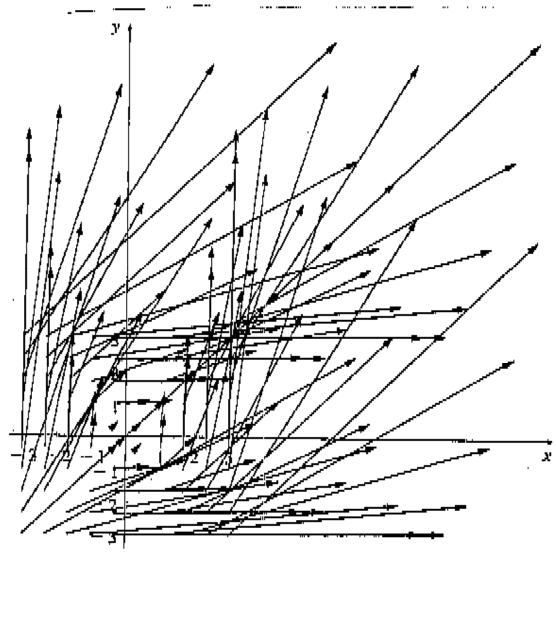
(C) $\frac{1}{\sqrt{x^2+y^2}}\langle x, y \rangle$

(D) $\langle y^2, x^2 \rangle$

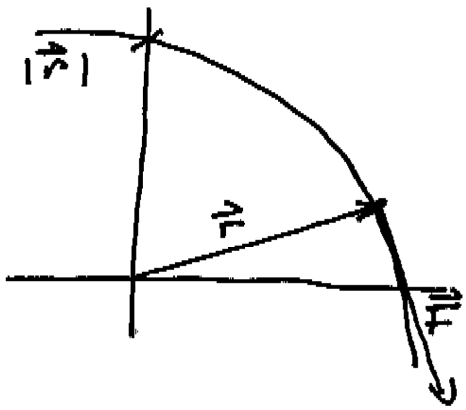








Ex 2: If $\vec{r} = \langle x, y \rangle$ and $\vec{F}(\vec{r}) = \langle y, -x \rangle$ then $\vec{r} \cdot \vec{F} = 0$ .... so $\vec{F}$ is always $\perp$ to its position vector $\vec{r}$... imagine a circle w/ radius $|\vec{r}|$ ....



$\vec{F}$ is $\parallel$ to the tangent vector ... and its magnitude is

$$|\vec{F}| = \sqrt{(y)^2 + (-x)^2}$$
$$= \sqrt{x^2 + y^2}$$
$= |\vec{r}|$ the same as the radius.

One vector field we have seen in the past is the gradient field. In $\mathbb{R}^2$

$$f : (x, y) \longmapsto z$$
$$\nabla f : (x, y) \longmapsto \langle f_x(x, y), f_y(x, y) \rangle$$

Ex 3: Plot the gradient field & contour plot together...

(a) $f(x, y) = \sin x + \sin y$

(b) $g(x, y) = \sin(x, y)$

Q: what happens when you go w/ the gradient? Against ... perpendicular

In order to set up a future application (in 16.3), recall from physics that Newton's Law of Gravitation states that the magnitude of the gravitational force between 2 objects w/ masses m & M is

$$\underbrace{|F| = \frac{mMG}{|\vec{r}|^2}}_{\text{scalar.}}$$

... G a gravitational constant

$|\vec{r}|$ is the dist. betw m & M.

If M is @ the origin, then the force $\vec{F}$ is in the direction of $-\vec{r}$.
and we can write $\vec{F} = \frac{mMG}{|\vec{r}|^2} \cdot \frac{-\vec{r}}{|\vec{r}|}$

$$= -\frac{mMG}{r^3} \cdot \vec{r}$$

$\vec{r} = \langle x, y, z \rangle$ (a position vector)

so $|\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$

AND

$$\vec{F}(x,y,z) = \frac{-mMG}{(x^2+y^2+z^2)^{3/2}} x\,\vec{i} + \frac{-mMG}{(x^2+y^2+z^2)^{3/2}} y\,\vec{j} + \frac{-mMG}{(x^2+y^2+z^2)^{3/2}} z\,\vec{k}$$

Now, if $f(x,y,z) = \frac{mMG}{\sqrt{x^2+y^2+z^2}}$, then $\nabla f = \vec{F}$.

... when $\exists f$ s.t. $\nabla f = \vec{F}$, we call $\vec{F}$ a conservative vector field & f its potential fct.

Q: where have you heard the words conservative & potential?