15.5: Apps of Double Integrals.

(I) Physics

(II) Probability — skip " #27 & #30 become bonus problems.

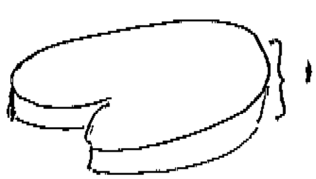
NOTE: Density is measured in mass/unit volume.

If $\rho(x,y)$ gives the density of a lamina at a pt (x,y), why does $\rho(x,y)\,dA$ give the mass of an infinitesimal rectangular region?

So, the mass is $m = \iint_D \rho(x,y)\,dA$

Ex1: Find the mass of a cylinder w/ density $\rho = 1$ over a base enclosed by $r = 1+\cos\theta$

picture a box of chocolates . . . instead of a lamina w/variable ~~thick~~ density, it is a solid w/ variable ht



$$m = 2\int_0^{\pi}\int_0^{1+\cos\theta} 1 \cdot r\,dr\,d\theta$$

$\vdots$

$$m = \frac{3}{2}\pi$$

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The moment of a particle is the product of its mass & distance from the axis.

$$M_x = \iint_D \overbrace{y}^{\text{dist}} \overbrace{\rho(x,y)\,dA}^{\text{mass}} \quad \text{AND} \quad M_y = \iint_D x\,\rho(x,y)\,dA$$

Ex/cont.: Find M_x & M_y.

(a) Find M_x.

$$= \int_0^{2\pi}\int_0^{1+\cos\theta} y \cdot 1 \; r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{1+\cos\theta} r^2 \sin\theta\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^3}{3}\sin\theta\right]_0^{1+\cos\theta} d\theta$$

$$= \int_0^{2\pi} \frac{(1+\cos\theta)^3}{3}\sin\theta\,d\theta$$

This is not an even fct and so the integral can't be simplified into $2\int_0^{\pi} \text{fct}\cdot d\theta$

Let $u = 1+\cos\theta$

$-du = \sin\theta\,d\theta$

$u(0) = 2$ & $u(2\pi) = 2$

As a matter of fact, the graph is "odd" about $\theta = \pi$.

$$= -\int_2^2 \frac{u^3}{3}\,du$$

$$= 0$$

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(b) Find M_y

$$= \int_0^{2\pi}\int_0^{1+\cos\theta} x \cdot 1\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{1+\cos\theta} r^2 \cos\theta \, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^3}{3}\cos\theta\right]_0^{1+\cos\theta} d\theta$$

$$= \int_0^{2\pi} \frac{(1+\cos\theta)^3}{3}\cos\theta \, d\theta$$

The integrand is symmetric about $\theta = \pi$, so can be $2\int_0^{\pi} f(\theta)\, d\theta$

$$= \frac{5\pi}{4}$$

(I confess, I used mathematica ... you can show the details for a HW problem ...)

And the center of mass is at $(\bar{x}, \bar{y})$

where $\bar{x} = \dfrac{M_y}{m}$ and $\bar{y} = \dfrac{M_x}{m}$

Ex1 cont: Find $(\bar{x}, \bar{y})$

recall $m = \frac{3}{2}\pi$; $M_x = 0$; $M_y = \frac{5\pi}{4}$

$\bar{x} = \frac{5\pi}{4} \cdot \frac{2}{3\pi} = \frac{5}{6}$ and $\bar{y} = \dfrac{0}{\frac{3}{2}\pi} = 0$

so $(\bar{x}, \bar{y}) = \left(\frac{5}{6}, 0\right)$. This makes sense given the picture

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Show[PolarPlot[1 + Cos[θ], {θ, 0, 2 π}, AspectRatio → 1],
  Graphics[{PointSize → Large, Point[{5/6, 0}]}], Ticks → None, AxesLabel → {x, y}]
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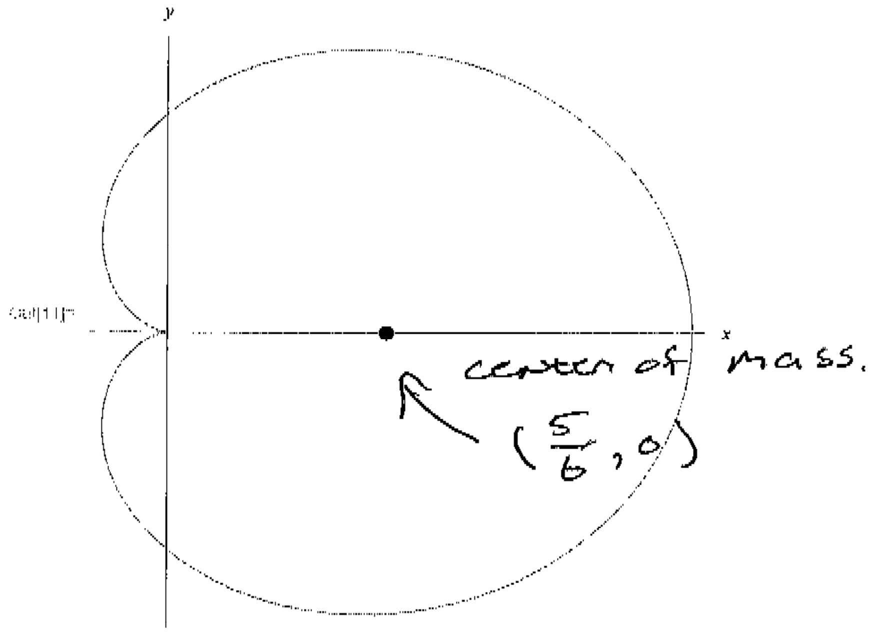

note: we could determine $\bar{y} = 0$ in this case by symmetry. But, if ρ had been non-constant, then ... we would have had to work.

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The moment of inertia is calculated in a similar manner.

$$I_x = \iint_D y^2 \rho(x,y)\, dA$$

$$I_y = \iint_D x^2 \rho(x,y)\, dA$$

AND $I_0 = I_x + I_y$ (moment of inertia about the origin).

Ex 2: A lamina w/ const density $\rho(x,y) = \rho$ occupies the region under $y = \sin x$ on $[0,\pi]$. Find the moments of inertia I_x & I_y

$$I_x = \int_0^{\pi}\int_0^{\sin x} y^2 \rho\, dy\, dx = \frac{4}{9}\rho \quad \text{(use substitution)}$$

$$I_y = \int_0^{\pi}\int_0^{\sin x} x^2 \rho\, dy\, dx = (\pi^2 - 4)\rho \quad \text{(use parts twice)}$$

The radius of gyration wRT the x-axis $\bar{\bar{y}}$

y-axis $\bar{\bar{x}}$

$$m\bar{\bar{y}}^2 = I_x \quad \& \quad m\bar{\bar{x}}^2 = I_y.$$