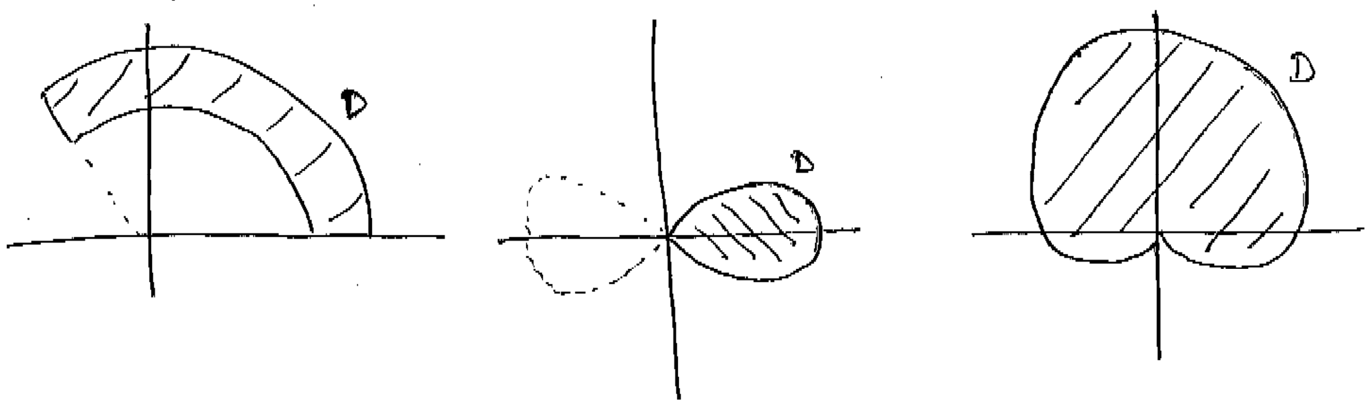
15.4 Double Integrals in Polar Coordinates

Why polar coordinates? Sometimes regions are easier to express this way



Finding the area of polar regions (10.4)

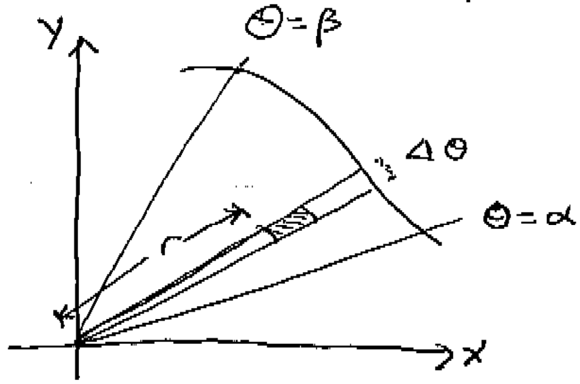
$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

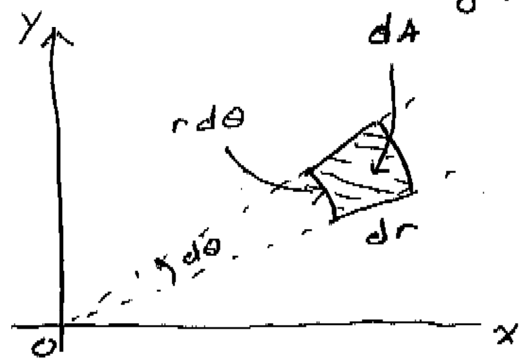
Now, find the volume of a solid over a region describe using polar coordinates.

$$\iint_D f(x,y)\, dA$$

$x = r\cos\theta$; $y = r\sin\theta$; $r^2 = x^2 + y^2$

This lets us describe D & f in polar coordinates, But, what about $dA = dx \cdot dy$?





is this a rectangle? — no, but in the limit we can see this as a rectangle.

So: Double integrals: Rectangular $\mapsto$ Polar

If f is continuous on the polar rectangle R given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Ex1: Derive the formula for the volume of a sphere of radius a.

$$\underbrace{x^2 + y^2}_{r^2} + z^2 = a^2$$

$$\Rightarrow \quad z^2 = a^2 - r^2$$

$$\Rightarrow \quad z = \pm\sqrt{a^2 - r^2}$$

we will use symmetry and integrate over the region R where $0 \le \theta \le \frac{\pi}{2}$ and $0 \le r \le a$

so
$$V = 8\iint_R z\,dA$$

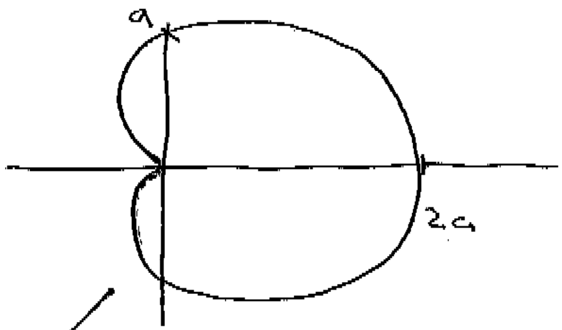
$$= 8\int_0^{\pi/2}\int_0^a \sqrt{a^2 - r^2}\, r\,dr\,d\theta \qquad \text{Let } u = a^2 - r^2, \quad \frac{du}{-2} = r\,dr$$

$$= -4\int_0^{\pi/2}\left[\frac{2}{3}u^{3/2}\right]_{a^2}^{0} d\theta$$

$$= +4\int_0^{\pi/2} \frac{2}{3}a^3\,d\theta$$

$$= \left[\frac{8}{3}a^3\theta\right]_0^{\pi/2} \qquad \frac{4\pi}{3}a^3.$$

Ex 3: Find the area of the cardiod $r = a(1 + \cos\theta)$.



The magnitude of the area is equal to that of a solid w/height 1 w/the same base (although the units differ).

Area "equal" to volume.

$$A = 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} 1 \; r\,dr\,d\theta$$

$$= 2\int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= 2\int_0^{\pi}\frac{a^2}{2}(1+\cos\theta)^2\,d\theta$$

$$= a^2\int_0^{\pi}(1 + 2\cos\theta + \cos^2\theta)\,d\theta$$

$$= a^2\int_0^{\pi}\left(1 + 2\cos\theta + \frac{1}{2}(1+\cos 2\theta)\right)d\theta$$

$$= a^2\left[\theta + 2\sin\theta + \frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right]_0^{\pi}$$

$$= a^2\left[\frac{3\pi}{2}\right]$$

15.4
3/4

Ex 2: The normal distribution is related to $\int_0^\infty e^{-x^2}\,dx$ (bell curve) ...

$$I = \int_0^\infty e^{-x^2}\,dx$$

$$I^2 = \int_0^\infty e^{-x^2}\,dx \int_0^\infty e^{-y^2}\,dy$$

$$= \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy.$$

can we do this w/ improper integrals?

$$= \int_0^{\pi/2}\int_0^\infty e^{-r^2}\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[-\frac{1}{2}e^{-r^2}\right]_0^\infty d\theta$$

$$= \int_0^{\pi/2}\frac{1}{2}\,d\theta$$

$$= \frac{\pi}{4}$$

so $I^2 = \frac{\pi}{4}$ AND $I = \frac{\sqrt{\pi}}{2}$

Story: Lord Kelvin on $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$

*There is a famous story about the nineteenth-century Scottish physicist Lord Kelvin. "Do you know what a mathematician is?" Kelvin once asked a class. He stepped to the blackboard and wrote

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi},$$

which is clearly equivalent to (4). "A mathematician," he continued, "is one to whom *that* is as obvious as twice two makes four is to you." As a matter of fact, this formula is *not* obvious, either to the present writer or to any of the many mathematicians he has known. The conclusion seems to be that Kelvin was both showing off and trying to put down his class in a rather mean-spirited way.