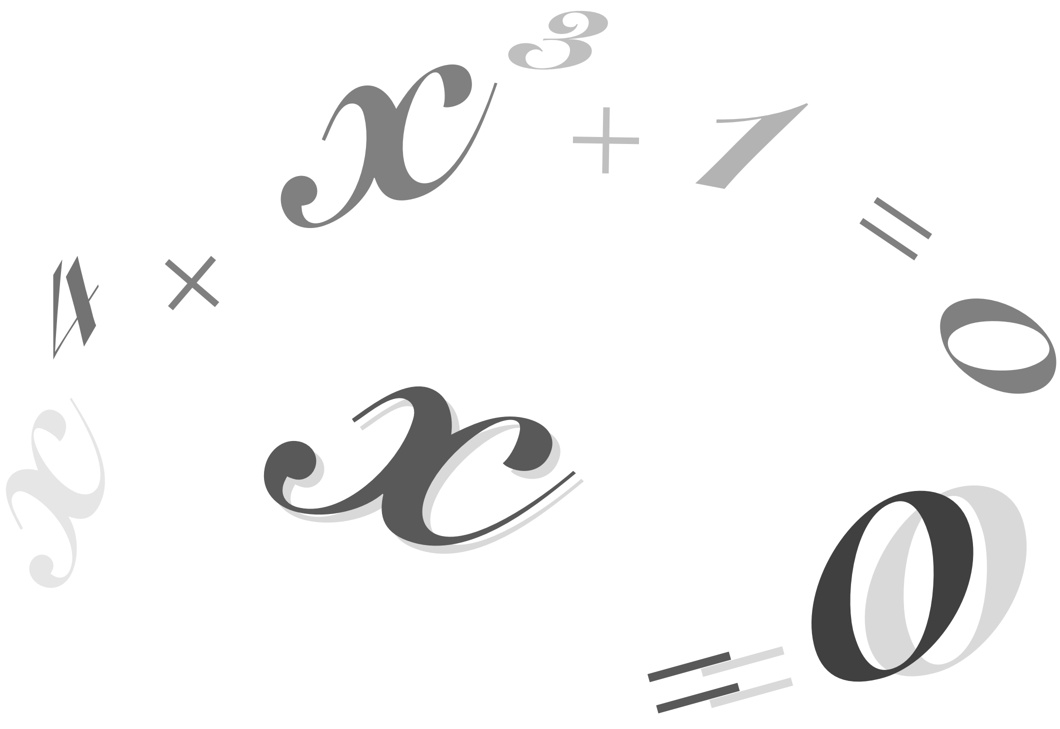
LESSON 13.1 – NONLINEAR EQUATIONS



OVERVIEW

Here's what you'll learn in this lesson:

Solving Equations

a. Solving polynomial equations by factoring

b. Solving quadratic type equations by factoring or by substitution

Radical Equations

a. Solving $\sqrt{ax + b} = cx + d$

b. Solving $\sqrt{ax + b} + \sqrt{cx + d} = ex + f$

c. Solving $\sqrt[n]{ax + b} = \sqrt[n]{cx + d}$

d. Solving equations that contain rational exponents

Suppose you were the owner of a hot dog cart. Of course you would be interested in knowing how many hot dogs you would need to sell each day to make the most profit. One way to figure this out would be to solve a profit equation based on your costs and pricing strategy. Most likely, such an equation would be nonlinear.

In this lesson, you will learn some techniques for solving nonlinear equations. First, you will learn to solve a variety of polynomial equations by factoring. Then you will learn to solve some equations that resemble quadratic equations. Finally, you will learn to solve equations in which variables appear under radical signs.

EXPLAIN

SOLVING EQUATIONS

Summary

Solving Nonlinear Equations by Factoring

Sometimes you will need to solve nonlinear equations. You can solve some nonlinear equations by rewriting them so they look like polynomial equations, then factoring.

To solve such a nonlinear equation by factoring:

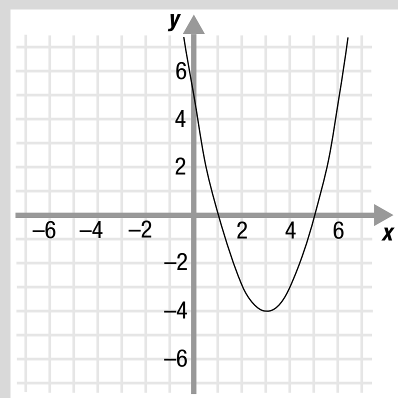
1. Write the equation as a polynomial equation in standard form.
2. Factor the left side.
3. Use the Zero Product Property to set each factor equal to 0.
4. Finish solving for the variable.

A polynomial equation is in standard form when the terms are arranged in descending order by degree on the left side of the equation and the right side equals 0.

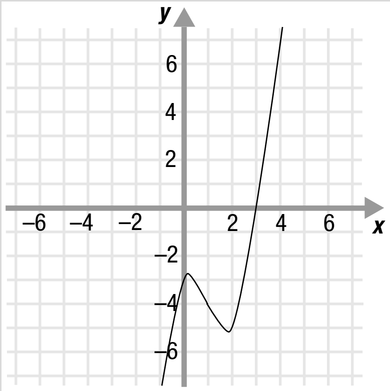
For example, to solve $x - 6 = -\frac{5}{x}$:

1. Write the equation in standard form. $$x \cdot (x-6) = x \cdot \left(-\frac{5}{x}\right)$$ $$x^2 - 6x = -5$$ $$x^2 - 6x + 5 = 0$$
2. Factor the left side. $$(x-1)(x-5) = 0$$
3. Use the Zero Product Property. $$x - 1 = 0 \text{ or } x - 5 = 0$$
4. Finish solving for *x*. $$x = 1 \text{ or } x = 5$$

So the two solutions of $x - 6 = -\frac{5}{x}$ are $x = 1$ and $x = 5$.



The graph of the function $y = x^2 - 6x + 5$ crosses the x-axis at $x = 1$ and $x = 5$, the two solutions of $x - 6 = -\frac{5}{x}$.



The graph of $y = x^3 - 3x^2 + x - 3$ only crosses the x-axis in one place because only one solution of the equation, $x^3 - 3x^2 + x = 3$, is a real number.

As another example, to solve $x^3 - 3x^2 + x = 3$:

1. Write the equation in standard form. $\quad x^3 - 3x^2 + x - 3 = 0$
2. Factor the left side. $\quad (x^3 - 3x^2) + (x - 3) = 0$

$$x^2(x - 3) + (x - 3) = 0$$

$$(x - 3)(x^2 + 1) = 0$$

3. Use the Zero Product Property. $\quad x - 3 = 0 \text{ or } x^2 + 1 = 0$
4. Finish solving for x. $\quad x = 3 \text{ or } x^2 = -1$

$$x = \pm\sqrt{-1}$$

$$x = \pm i$$

So the three solutions of $x^3 - 3x^2 + x = 3$ are $x = 3$, $x = -i$, and $x = i$.

Solving Nonlinear Equations by Substitution

The equations you looked at in the previous section could be solved by factoring. But sometimes when a nonlinear equation resembles a quadratic equation, it is easier to first use substitution to rewrite the problem before you factor it.

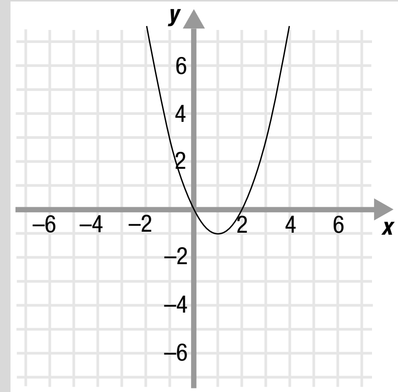
To solve such a polynomial equation using substitution:

1. Write the polynomial equation so it has the form of a quadratic equation.
2. Use an appropriate substitution to make the polynomial equation easier.
3. Factor the new equation.
4. Use the Zero Product Property to set each factor equal to 0.
5. Finish solving for the new variable.
6. Substitute the original expression back for the new variable.
7. Finish solving for the original variable.

For example, to solve $(x-4)^2 + 6(x-4) = -8$:

1. Write the polynomial so it is quadratic in form. $(x-4)^2 + 6(x-4) + 8 = 0$
2. Substitute u for $(x-4)$. $u^2 + 6u + 8 = 0$
3. Factor. $(u+4)(u+2) = 0$
4. Use the Zero Product Property. $u + 4 = 0$ or $u + 2 = 0$
5. Finish solving for u. $u = -4$ or $u = -2$
6. Substitute $(x-4)$ for u. $x - 4 = -4$ or $x - 4 = -2$
7. Finish solving for x. $x = 0$ or $x = 2$

The solutions, $x = 0$ and $x = 2$, both satisfy the original equation, $(x-4)^2 + 6(x-4) = -8$.



The graph of the function $y = (x-4)^2 + 6(x-4) + 8$ crosses the x-axis at $x = 0$ and $x = 2$, the two solutions of the equation, $(x-4)^2 + 6(x-4) = -8$.

Sample Problems

1. Solve for x: $x + 7 + \frac{10}{x} = 0$.

✔ a. Write the equation in standard form.

$$x + 7 + \frac{10}{x} = 0$$

$$x \cdot \left(x + 7 + \frac{10}{x}\right) = x \cdot 0$$

$$x^2 + 7x + 10 = 0$$

☐ b. Factor the left side. __________ $= 0$

☐ c. Use the Zero Product Property. ______ $= 0$ or ______ $= 0$

☐ d. Finish solving for x. $x =$ ______ or $x =$ ______

2. Use substitution to solve for y: $(y-2)^2 + 10(y-2) + 9 = 0$.

✔ a. Substitute u for $(y-2)$. $u^2 + 10u + 9 = 0$

☐ b. Factor. __________ $= 0$

☐ c. Use the Zero Product Property. ______ $= 0$ or ______ $= 0$

☐ d. Finish solving for u. $u =$ ______ or $u =$ ______

☐ e. Substitute $(y-2)$ for u. $y - 2 =$ ____ or $y - 2 =$ ____

☐ f. Finish solving for y. $y =$ ______ or $y =$ ______

Answers to Sample Problems

b. $(x+2)(x+5)$

c. $x + 2$, $x + 5$

d. -2, -5

b. $(u+1)(u+9)$

c. $u + 1$, $u + 9$

d. -1, -9

e. -1, -9

f. 1, -7

RADICAL EQUATIONS

Summary

Equations with Radicals

Sometimes in an equation the variable is under a radical sign. When this is the case you need to get the variable out from under the radical sign in order to solve the equation.

If an equation has variables that appear under two radical signs, separate the radicals on opposite sides of the equation.

For example, in step (2), if the index of the radical is 2, square both sides of the equation. If the index of the radical is 3, cube both sides of the equation.

To solve an equation in which the variable is under a radical sign:

1. Isolate a radical on one side of the equation.
2. Raise both sides of the equation to the same power as the index of the isolated radical. Repeat steps (1) and (2) as often as necessary to eliminate all radicals.
3. Rewrite the resulting equation as a polynomial equation in standard form.
4. Factor.
5. Solve for the variable.
6. Check your answer in the original equation.

It is very important to check your answer in the original equation. It is possible to get answers that don't make sense.

For example, to solve $x + 1 = \sqrt{x + 3}$:

1. Isolate a radical on one side of the equation. $\sqrt{x + 3}$ is already isolated
2. Square both sides of the equation. $(x + 1)^2 = \left(\sqrt{x + 3}\right)^2$
   $x^2 + 2x + 1 = x + 3$
3. Rewrite the equation in standard form. $x^2 + x - 2 = 0$
4. Factor. $(x + 2)(x - 1) = 0$
5. Solve for x. $x + 2 = 0$ or $x - 1 = 0$
   $x = -2$ or $x = 1$
6. Check your answer.

Check $x = -2$:	Check $x = 1$:
Is $(-2) + 1 = \sqrt{-2 + 3}$?	Is $1 + 1 = \sqrt{1 + 3}$?
Is $-1 = \sqrt{1}$?	Is $2 = \sqrt{4}$?
Is $-1 = 1$? No.	Is $2 = 2$? Yes.

$x = -2$ is an extraneous solution because it does not satisfy the original equation.

So the only solution of the equation $x + 1 = \sqrt{x + 3}$ is $x = 1$.

As another example, to solve $\sqrt{x+5} - \sqrt{x-3} = 2$:

1. Isolate a radical on one side of the equation. $\sqrt{x+5} = \sqrt{x-3} + 2$

2. Square both sides of the equation.
$$\left(\sqrt{x+5}\right)^2 = \left(\sqrt{x-3} + 2\right)^2$$
$$x + 5 = (x - 3) + 4\sqrt{x-3} + 4$$
$$x + 5 = x + 1 + 4\sqrt{x-3}$$

3. Isolate a radical on one side of the equation. $4 = 4\sqrt{x-3}$

4. Square both sides of the equation. $4^2 = \left(4\sqrt{x-3}\right)^2$

5. Rewrite the equation so it is a polynomial equation. $16 = 16(x - 3)$

6. Solve for x.
$$1 = x - 3$$
$$4 = x$$

7. Check your answer. Check $x = 4$:

   Is $\sqrt{4+5} - \sqrt{4-3} = 2$?

   Is $\sqrt{9} - \sqrt{1} = 2$?

   Is $3 - 1 = 2$?

   Is $2 = 2$? Yes.

So $x = 4$ is the solution of the equation $\sqrt{x+5} - \sqrt{x-3} = 2$.

Equations with Rational Exponents

Equations that contain rational exponents are solved in much the same way as equations that contain radical expressions.

To solve an equation that contains a rational exponent:

1. Isolate the term that contains the rational exponent on one side of the equation.
2. Rewrite the equation using a radical.
3. Finish solving for the variable.
4. Check your answer in the original equation.

For example, to solve the equation $(x+1)^{\frac{3}{2}} - 5 = -1$:

1. Isolate the term with the rational exponent. $\quad (x+1)^{\frac{3}{2}} = 4$

2. Rewrite the equation using a radical. $\quad \sqrt{(x+1)^3} = 4$

3. Finish solving for x.

$$\left(\sqrt{(x+1)^3}\right)^2 = (4)^2$$
$$(x+1)^3 = 16$$
$$\sqrt[3]{(x+1)^3} = \sqrt[3]{16}$$
$$x+1 = \sqrt[3]{2^3 \cdot 2}$$
$$x+1 = 2\sqrt[3]{2}$$
$$x = 2\sqrt[3]{2} - 1$$

4. Check your answer. Check $x = 2\sqrt[3]{2} - 1$:

Is $\left[(2\sqrt[3]{2} - 1) + 1\right]^{\frac{3}{2}} - 5 = -1$?

Is $\left(2\sqrt[3]{2}\right)^{\frac{3}{2}} - 5 = -1$?

Is $\left(2 \cdot 2^{\frac{1}{3}}\right)^{\frac{3}{2}} - 5 = -1$?

Is $\left(2^3 \cdot 2^{\frac{1}{3} \cdot 3}\right)^{\frac{1}{2}} - 5 = -1$?

Is $(8 \cdot 2)^{\frac{1}{2}} - 5 = -1$?

Is $(16)^{\frac{1}{2}} - 5 = -1$?

Is $\sqrt{16} - 5 = -1$?

Is $4 - 5 = -1$?

Is $-1 = -1$? Yes.

So $x = 2\sqrt[3]{2} - 1$ is the solution of the equation $(x+1)^{\frac{3}{2}} - 5 = -1$.

Sample Problems

1. Solve for x: $(x-61)^{\frac{3}{5}} + 34 = 7$.

✔ a. Isolate the term with the rational exponent. $\quad (x-61)^{\frac{3}{5}} = -27$

✔ b. Rewrite the equation using a radical. $\quad \sqrt[5]{(x-61)^3} = -27$

☐ c. Finish solving for x.

$$\left(\sqrt[5]{(x-61)^3}\right)^5 = (-27)^5$$
$$(x-61)^3 = (-27)^5$$
$$\sqrt[3]{(x-61)^3} = \left(\sqrt[3]{-27}\right)^5$$
$$x-61 = \left(\sqrt[3]{-27}\right)^5$$
$$x-61 = (-3)^5$$
$$x-61 = \_\_\_\_\_$$
$$x = \_\_\_\_\_\_$$

☐ d. Check your answer.

2. Solve for x: $(x+6)^{\frac{2}{5}} + 2 = 3$.

✔ a. Isolate the term with the rational exponent. $\quad (x+6)^{\frac{2}{5}} = 1$

✔ b. Rewrite the equation using a radical. $\quad \sqrt[5]{(x+6)^2} = 1$

☐ c. Finish solving for x.

$$\left(\sqrt[5]{(x+6)^2}\right)^5 = (1)^5$$
$$(x+6)^2 = 1$$
$$x+6 = \_\_\_\_\_\_ \text{ or } x+6 = \_\_\_\_\_\_$$
$$x = \_\_\_\_\_ \quad \text{or} \quad x = \_\_\_\_\_$$

☐ d. Check your answer.

Answers to Sample Problems

c. –243

–182

d. Here's one way to check:

Check $x = -182$:

$$\text{Is } (-182-61)^{\frac{3}{5}} + 34 = 7?$$
$$\text{Is } (-243)^{\frac{3}{5}} + 34 = 7?$$
$$\text{Is } \left(\sqrt[5]{-243}\right)^3 + 34 = 7?$$
$$\text{Is } (-3)^3 + 34 = 7?$$
$$\text{Is } -27 + 34 = 7?$$
$$\text{Is } 7 = 7? \text{ Yes.}$$

c. 1, –1

–5, –7

d. Here's one way to check:

Check $x = -5$:

$$\text{Is } (-5+6)^{\frac{2}{5}} + 2 = 3?$$
$$\text{Is } (1)^{\frac{2}{5}} + 2 = 3?$$
$$\text{Is } \left(\sqrt[5]{1}\right)^2 + 2 = 3?$$
$$\text{Is } 1^2 + 2 = 3?$$
$$\text{Is } 1 + 2 = 3?$$
$$\text{Is } 3 = 3? \text{ Yes.}$$

Check $x = -7$:

$$\text{Is } (-7+6)^{\frac{2}{5}} + 2 = 3?$$
$$\text{Is } (-1)^{\frac{2}{5}} + 2 = 3?$$
$$\text{Is } \left(\sqrt[5]{-1}\right)^2 + 2 = 3?$$
$$\text{Is } (-1)^2 + 2 = 3?$$
$$\text{Is } 1 + 2 = 3?$$
$$\text{Is } 3 = 3? \text{ Yes.}$$

EXPLORE

Sample Problems

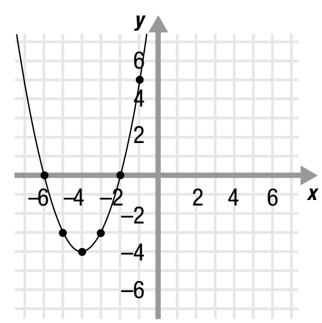
On the computer you used the Grapher to investigate the real solutions of nonlinear equations. Below are some additional exploration problems.

1. Graph the parabola $y = x^2 + 8x + 12$. Use your graph to find the solutions of the quadratic equation $x^2 + 8x + 12 = 0$.

☑ a. Graph the parabola $y = x^2 + 8x + 12$. Make a table of values.

x	y
–1	5
–2	0
–3	–3
–4	–4
–5	–3
–6	0

Plot these points on a grid and graph the parabola.



☑ b. Find the points where the graph of the parabola crosses the x-axis. The x-coordinates of these points are the solutions of the equation.

The graph crosses the x-axis at (–2, 0) and (–6, 0). So the solutions of $x^2 + 8x + 12 = 0$ are $x = -2$ and $x = -6$.

☐ c. Check your answer using algebra.

Answers to Sample Problems

c. Here's one way to check:

Check $x = -2$:
Is $(-2)^2 + 8(-2) + 12 = 0$?
Is $4 - 16 + 12 = 0$?
Is $0 = 0$? Yes.

Check $x = -6$:
Is $(-6)^2 + 8(-6) + 12 = 0$?
Is $36 - 48 + 12 = 0$?
Is $0 = 0$? Yes.

2. Solve the equation $x^2 + 3x = 4$ by graphing $y = x^2 + 3x$ and $y = 4$.

☐ a. Graph the parabola $y = x^2 + 3x$. Make a table of values.

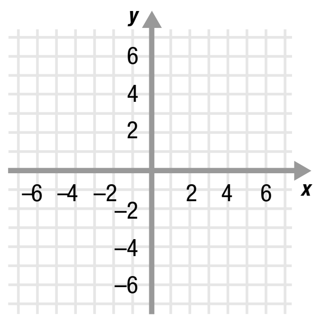
x	y
1	4
0	0
–1	–2
–1.5	–2.25
–2	–2
–3	0
–4	4

Plot these points on the grid below and graph the parabola.

☐ b. Graph the line $y = 4$. Make a table of values.

x	y
–3	4
0	4
2	4

Plot these points on the grid and graph the line.



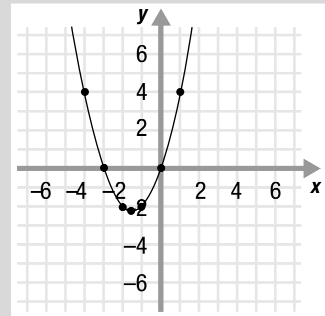
☐ c. Find where the graphs cross. The x-coordinates of the points of intersection are the solutions of the equation $x^2 + 3x = 4$.

The two graphs cross at (____, ____) and (____, ____), so the solutions of the quadratic equation $x^2 + 3x = 4$ are $x =$ _____ or $x =$ _____.

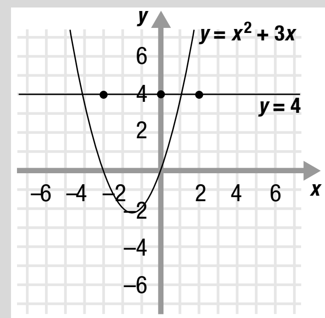
☐ d. Check your answer using algebra.

Answers to Sample Problems

a.



b.



c. (–4, 4) and (1, 4)

–4

1

d. Here's one way to check:

Check $x = –4$:
Is $0 = (–4)^2 + 3(–4) – 4$?
Is $0 = \ 16 \ – \ 12 \ – 4$?
Is $0 = 0$? Yes.

Check $x = 1$:
Is $0 = 1^2 + 3(1) – 4$?
Is $0 = 1 \ + \ 3 \ – 4$?
Is $0 = 0$? Yes.

HOMEWORK

Homework Problems

Circle the homework problems assigned to you by the computer, then complete them below.

Explain

Solving Equations

1. Solve for x: $x^4 - 7x^2 + 12 = 0$
2. Use substitution to solve for y: $y^4 - 7y^2 + 12 = 0$
3. Solve for w: $2w^3 - w^2 + 6w - 3 = 0$
4. Use substitution to solve for x: $x^{-2} + x^{-1} = 20$
5. Solve for a: $\frac{a}{2} - 3 = -\frac{4}{a}$
6. Use substitution to solve for y: $y^4 - 17y^2 + 72 = 0$
7. Solve for b: $b^4 - 8b^3 + 15b^2 = 0$
8. Use substitution to solve for z: $\left(z + \frac{6}{z}\right)^2 + 12\left(z + \frac{6}{z}\right) + 35 = 0$
9. Solve for x: $x - \frac{3}{2x} = \frac{1}{2}$
10. Use substitution to solve for w: $(w - 6)^4 - 81 = 0$
11. Solve for p: $p + \frac{2}{p} = \frac{19}{3}$
12. Use substitution to solve for r: $\left(r + \frac{4}{r}\right)^2 = 5\left(r + \frac{4}{r}\right)$

Radical Equations

13. Solve for x: $\sqrt{2x + 5} = x + 3$
14. Solve for x: $(x - 5)^{\frac{1}{2}} + 4 = 6$
15. Solve for x: $x + 6 = \sqrt{x + 6}$
16. Solve for x: $\sqrt{1 - x} = x + 5$
17. Solve for x: $(x + 9)^{\frac{3}{4}} + 2 = 10$
18. Solve for x: $x + 2 = \sqrt[3]{12x + 8}$
19. Solve for x: $(x + 26)^{\frac{3}{5}} = 27$
20. Solve for x: $\sqrt{3x + 4} - \sqrt{2x + 1} = 1$
21. Solve for x: $\sqrt[4]{x^2 - 8x - 4} - 2 = 0$
22. Solve for x: $(x - 3)^{\frac{2}{3}} = 5$
23. Solve for x: $\sqrt{3(x + 4)} = 1 + \sqrt{2x + 7}$
24. Solve for x: $(x - 11)^{\frac{5}{2}} - 5 = 27$

Explore

25. Graph the parabola $y = x^2 + 3x + 2$. Use your graph to find the solutions of the quadratic equation $x^2 + 3x + 2 = 0$.
26. For what value of c does the equation $x^2 + 12x + c = 0$ have only one distinct solution?
27. Solve the equation $x^2 + 5x = -6$ by graphing $y = x^2 + 5x$ and $y = -6$.
28. Graph the parabola $y = x^2 + 3x - 10$. Use your graph to find the solutions of the quadratic equation $x^2 + 3x - 10 = 0$.
29. For what values of c does the equation $x^2 + c = 0$ have at least one real solution?
30. Solve the equation $x^2 = x + 2$ by graphing $y = x^2$ and $y = x + 2$.

APPLY

Practice Problems

Here are some additional practice problems for you to try.

1. Solve for w: $w^4 - 15w^2 - 16 = 0$
2. Solve for t: $t^4 + 15 = 8t^2$
3. Use substitution to solve for w: $w^{-2} + 5w^{-1} - 36 = 0$
4. Use substitution to solve for y: $(y + 5)^2 - 4(y + 5) - 12 = 0$
5. Solve for x by grouping: $x^3 - 5x^2 - 9x + 45 = 0$
6. Solve for r by grouping: $3r^3 - 2r^2 - 21r + 14 = 0$
7. Solve for z: $z - \frac{6}{5z} = -\frac{7}{5}$
8. Solve for q: $\frac{q}{3} - 3 = \frac{12}{q}$
9. Solve for x: $x^4 - 8x^3 - 9x^2 = 0$
10. Solve for t: $t^{-2} - t^{-1} - 42 = 0$

11. Solve for x: $\sqrt{4x + 21} = 3$

12. Solve for x: $\sqrt{9x - 2} = x + 2$

18. Solve for x: $x + 2 = \sqrt[3]{12x + 8}$

14. Solve for x: $x - 9 = \sqrt{x - 9}$

15. Solve for x: $(x - 1)^{\frac{3}{2}} - 1 = 7$

16. Solve for x: $(5x - 3)^{\frac{1}{3}} + 8 = 11$

17. Solve for x: $\sqrt[4]{x^2 - 4x + 11} + 6 = 8$

18. Solve for x: $\sqrt[3]{x^2 - 7x - 15} + 3 = 0$

19. Solve for x: $\sqrt{x + 10} - \sqrt{x + 5} = 1$

20. Solve for x: $\sqrt{5 - x} + 3 = \sqrt{3x + 4}$

EVALUATE

Practice Test

Take this practice test to be sure that you are prepared for the final quiz in Evaluate.

1. Solve for x: $\frac{1}{x} - \frac{4}{3} = \frac{x}{4}$
2. What value of u can you substitute into the equation $5x^{-8} + 3x^{-4} - 2 = 0$ to get an equation in standard quadratic form?
3. Solve for y: $y^3 - 28 = 4y - 7y^2$
4. Solve for w: $\left(\frac{w-4}{w}\right)^2 = 5\left(\frac{w-4}{w}\right) - 6$
5. Solve for x: $\sqrt{x^2 + 6} - x = 3$
6. Solve for y: $\sqrt{y+8} - \sqrt{y+1} = 2$
7. Solve for x: $\sqrt[4]{x^3 + 54} - 2 = 1$
8. Solve for w: $(w+5)^{\frac{2}{3}} - 5 = -1$
9. Graph the parabola $y = x^2 - x - 2$. Use your graph to find the solutions of the quadratic equation $x^2 - x - 2 = 0$.
10. Graph the parabola $y = x^2 - 4x$ and the line $y = -3$. Use your graphs to find the solutions of the equation $x^2 - 4x = -3$.
11. The graphs of four parabolas are shown on the grid in Figure 13.1.1. Use these graphs to decide which of the equations below have one positive solution and one negative solution.

    $0 = x^2 + \frac{9}{2}x + 5$ (use graph a)

    $0 = \frac{1}{2}x^2 - \frac{5}{2}x - 3$ (use graph b)

    $0 = \frac{1}{3}x^2 - \frac{4}{3}x + 1$ (use graph c)

    $0 = x^2 - \frac{8}{3}x - 1$ (use graph d)

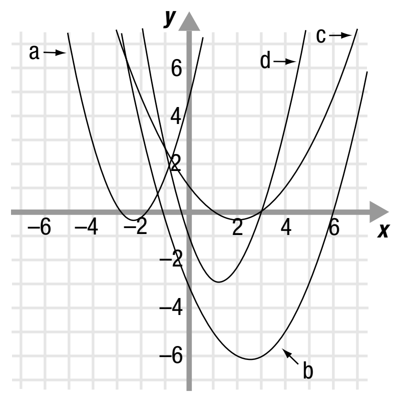


Figure 13.1.1

12. Graph the parabolas below. Use your graphs to determine for what values of c the equation $x^2 + 2x + c = 0$ has no solution.

    $y = x^2 + 2x - 3$

    $y = x^2 + 2x + 1$

    $y = x^2 + 2x + 4$