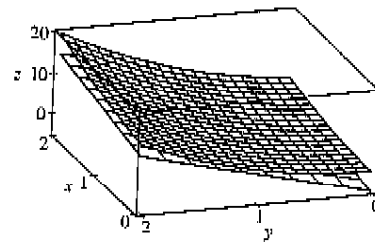
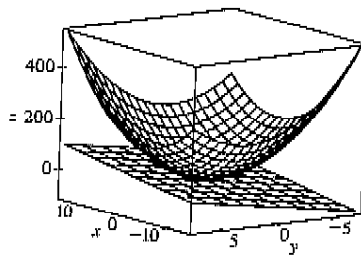


15.4 Tangent Planes and Linear Approximations

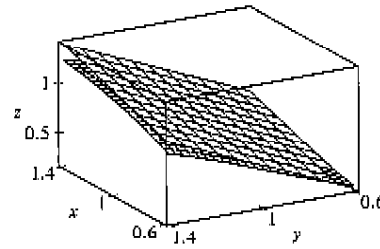
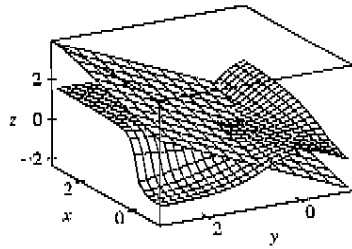
ET 14.4

1. $z = f(x, y) = 4x^2 - y^2 + 2y \Rightarrow f_x(x, y) = 8x, f_y(x, y) = -2y + 2$, so $f_x(-1, 2) = -8, f_y(-1, 2) = -2$.
By Equation 2, an equation of the tangent plane is $z - 4 = f_x(-1, 2)[x - (-1)] + f_y(-1, 2)(y - 2) \Rightarrow$
 $z - 4 = -8(x + 1) - 2(y - 2)$ or $z = -8x - 2y$.
2. $z = f(x, y) = 9x^2 + y^2 + 6x - 3y + 5 \Rightarrow f_x(x, y) = 18x + 6, f_y(x, y) = 2y - 3$, so $f_x(1, 2) = 24$ and
 $f_y(1, 2) = 1$. By Equation 2, an equation of the tangent plane is $z - 18 = f_x(1, 2)(x - 1) + f_y(1, 2)(y - 2) \Rightarrow$
 $z - 18 = 24(x - 1) + 1(y - 2)$ or $z = 24x + y - 8$.
3. $z = f(x, y) = \sqrt{4 - x^2 - 2y^2} \Rightarrow f_x(x, y) = \frac{1}{2}(4 - x^2 - 2y^2)^{-1/2}(-2x) = -\frac{x}{\sqrt{4 - x^2 - 2y^2}}$,
 $f_y(x, y) = \frac{1}{2}(4 - x^2 - 2y^2)^{-1/2}(-4y) = -\frac{2y}{\sqrt{4 - x^2 - 2y^2}}$, so $f_x(1, -1) = -1$ and $f_y(1, -1) = 2$. Thus, an
equation of the tangent plane is $z - 1 = f_x(1, -1)(x - 1) + f_y(1, -1)[y - (-1)] \Rightarrow$
 $z - 1 = -1(x - 1) + 2(y + 1)$ or $x - 2y + z = 4$.
4. $z = f(x, y) = y \ln x \Rightarrow f_x(x, y) = y/x, f_y(x, y) = \ln x$, so $f_x(1, 4) = 4, f_y(1, 4) = 0$, and an equation of
the tangent plane is $z - 0 = f_x(1, 4)(x - 1) + f_y(1, 4)(y - 4) \Rightarrow z = 4(x - 1) + 0(y - 4)$ or $z = 4x - 4$.
5. $z = f(x, y) = y \cos(x - y) \Rightarrow f_x = y(-\sin(x - y))(1) = -y \sin(x - y)$,
 $f_y = y(-\sin(x - y))(-1) + \cos(x - y) = y \sin(x - y) + \cos(x - y)$, so $f_x(2, 2) = -2 \sin(0) = 0$,
 $f_y(2, 2) = 2 \sin(0) + \cos(0) = 1$ and an equation of the tangent plane is $z - 2 = 0(x - 2) + 1(y - 2)$ or $z = y$.
6. $z = f(x, y) = e^{x^2 - y^2} \Rightarrow f_x(x, y) = 2xe^{x^2 - y^2}, f_y(x, y) = -2ye^{x^2 - y^2}$, so $f_x(1, -1) = 2, f_y(1, -1) = 2$.
By Equation 2, an equation of the tangent plane is $z - 1 = f_x(1, -1)(x - 1) + f_y(1, -1)[y - (-1)] \Rightarrow$
 $z - 1 = 2(x - 1) + 2(y + 1)$ or $z = 2x + 2y + 1$.
7. $z = f(x, y) = x^2 + xy + 3y^2$, so $f_x(x, y) = 2x + y \Rightarrow f_x(1, 1) = 3, f_y(x, y) = x + 6y \Rightarrow f_y(1, 1) = 7$
and an equation of the tangent plane is $z - 5 = 3(x - 1) + 7(y - 1)$ or $z = 3x + 7y - 5$. After zooming in, the
surface and the tangent plane become almost indistinguishable. (Here, the tangent plane is below the surface.) If we
zoom in farther, the surface and the tangent plane will appear to coincide.



8. $z = f(x, y) = \arctan(xy^2) \Rightarrow f_x = \frac{1}{1 + (xy^2)^2}(y^2) = \frac{y^2}{1 + x^2y^4}, f_y = \frac{1}{1 + (xy^2)^2}(2xy) = \frac{2xy}{1 + x^2y^4}$,
 $f_x(1, 1) = \frac{1}{1+1} = \frac{1}{2}, f_y(1, 1) = \frac{2}{1+1} = 1$, so an equation of the tangent plane is $z - \frac{\pi}{4} = \frac{1}{2}(x - 1) + 1(y - 1)$ or

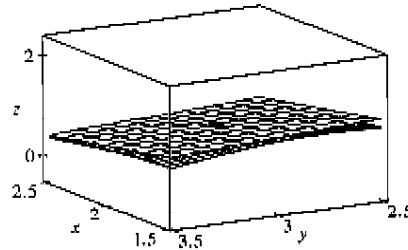
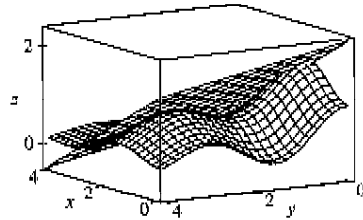
$z = \frac{1}{3}x + y - \frac{3}{2} + \frac{\pi}{4}$. After zooming in, the surface and the tangent plane become almost indistinguishable. (Here the tangent plane is above the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.



9. $f(x, y) = e^{-(x^2+y^2)/16}(\sin^2 x + \cos^2 y)$. A CAS gives

$$f_x = -\frac{2}{16}e^{-(x^2+y^2)/16}(x \sin^2 x + x \cos^2 y - 15 \sin x \cos x) \text{ and}$$

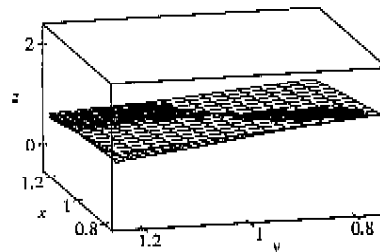
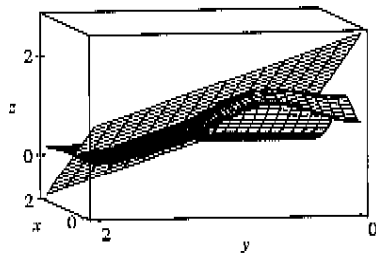
$f_y = -\frac{2}{16}e^{-(x^2+y^2)/16}(y \sin^2 x + y \cos^2 y + 15 \sin y \cos y)$. We use the CAS to evaluate these at $(2, 3)$, and then substitute the results into Equation 2 in order to plot the tangent plane. After zooming in, the surface and the tangent plane become almost indistinguishable. (Here, the tangent plane is above the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.



10. $f(x, y) = \frac{\sqrt{1+4x^2+4y^2}}{1+x^4+y^4}$. A CAS gives $f_x = \frac{4x(1-3x^4+y^4-x^2-4x^2y^2)}{\sqrt{1+4x^2+4y^2}(1+x^4+y^4)^2}$ and

$$f_y = \frac{4y(1-3y^4+x^4-y^2-4x^2y^2)}{\sqrt{1+4x^2+4y^2}(1+x^4+y^4)^2}.$$

We use the CAS to evaluate these at $(1, 1)$, and then substitute the results into Equation 2 to get an equation of the tangent plane: $z = \frac{25-8x-8y}{9}$. After zooming in, the surface and the tangent plane become almost indistinguishable. (Here, the tangent plane is shown with fewer traces than the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.



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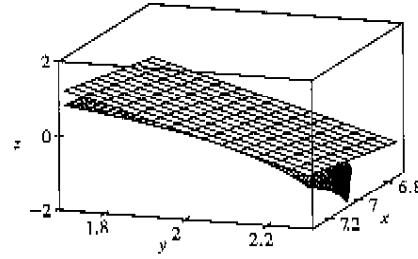
11. $f(x, y) = x\sqrt{y}$. The partial derivatives are $f_x(x, y) = \sqrt{y}$ and $f_y(x, y) = \frac{x}{2\sqrt{y}}$, so $f_x(1, 4) = 2$ and $f_y(1, 4) = \frac{1}{4}$. Both f_x and f_y are continuous functions for $y > 0$, so by Theorem 8, f is differentiable at $(1, 4)$. By Equation 3, the linearization of f at $(1, 4)$ is given by
- $$L(x, y) = f(1, 4) + f_x(1, 4)(x - 1) + f_y(1, 4)(y - 4) = 2 + 2(x - 1) + \frac{1}{4}(y - 4) = 2x + \frac{1}{4}y - 1.$$
12. $f(x, y) = \frac{x}{y}$. The partial derivatives are $f_x(x, y) = \frac{1}{y}$ and $f_y(x, y) = -\frac{x}{y^2}$, so $f_x(6, 3) = \frac{1}{3}$ and $f_y(6, 3) = -\frac{2}{3}$. Both f_x and f_y are continuous functions for $y \neq 0$, so f is differentiable at $(6, 3)$ by Theorem 8. The linearization of f at $(6, 3)$ is given by
- $$L(x, y) = f(6, 3) + f_x(6, 3)(x - 6) + f_y(6, 3)(y - 3) = 2 + \frac{1}{3}(x - 6) - \frac{2}{3}(y - 3) = \frac{1}{3}x - \frac{2}{3}y + 2.$$
13. $f(x, y) = e^x \cos xy$. The partial derivatives are $f_x(x, y) = e^x(\cos xy - y \sin xy)$ and $f_y(x, y) = -xe^x \sin xy$, so $f_x(0, 0) = 1$ and $f_y(0, 0) = 0$. Both f_x and f_y are continuous functions, so f is differentiable at $(0, 0)$ by Theorem 8. The linearization of f at $(0, 0)$ is given by
- $$L(x, y) = f(0, 0) + f_x(0, 0)(x - 0) + f_y(0, 0)(y - 0) = 1 + 1(x - 0) + 0(y - 0) = x + 1.$$
14. $f(x, y) = \sqrt{x + e^{4y}} = (x + e^{4y})^{1/2}$. The partial derivatives are $f_x(x, y) = \frac{1}{2}(x + e^{4y})^{-1/2}$ and $f_y(x, y) = \frac{1}{2}(x + e^{4y})^{-1/2}(4e^{4y}) = 2e^{4y}(x + e^{4y})^{-1/2}$, so $f_x(3, 0) = \frac{1}{2}(3 + e^0)^{-1/2} = \frac{1}{4}$ and $f_y(3, 0) = 2e^0(3 + e^0)^{-1/2} = 1$. Both f_x and f_y are continuous functions near $(3, 0)$, so f is differentiable at $(3, 0)$ by Theorem 8. The linearization of f at $(3, 0)$ is
- $$L(x, y) = f(3, 0) + f_x(3, 0)(x - 3) + f_y(3, 0)(y - 0) = 2 + \frac{1}{4}(x - 3) + 1(y - 0) = \frac{1}{4}x + y + \frac{5}{4}.$$
15. $f(x, y) = \tan^{-1}(x + 2y)$. The partial derivatives are $f_x(x, y) = \frac{1}{1 + (x + 2y)^2}$ and $f_y(x, y) = \frac{2}{1 + (x + 2y)^2}$, so $f_x(1, 0) = \frac{1}{2}$ and $f_y(1, 0) = 1$. Both f_x and f_y are continuous functions, so f is differentiable at $(1, 0)$, and the linearization of f at $(1, 0)$ is
- $$L(x, y) = f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0) = \frac{\pi}{4} + \frac{1}{2}(x - 1) + 1(y) = \frac{1}{2}x + y + \frac{\pi}{4} - \frac{1}{2}.$$
16. $f(x, y) = \sin(2x + 3y)$. The partial derivatives are $f_x(x, y) = 2\cos(2x + 3y)$ and $f_y(x, y) = 3\cos(2x + 3y)$, so $f_x(-3, 2) = 2$ and $f_y(-3, 2) = 3$. Both f_x and f_y are continuous functions, so f is differentiable at $(-3, 2)$, and the linearization of f at $(-3, 2)$ is
- $$L(x, y) = f(-3, 2) + f_x(-3, 2)(x + 3) + f_y(-3, 2)(y - 2) = 0 + 2(x + 3) + 3(y - 2) = 2x + 3y.$$
17. $f(x, y) = \sqrt{20 - x^2 - 7y^2} \Rightarrow f_x(x, y) = -\frac{x}{\sqrt{20 - x^2 - 7y^2}}$ and $f_y(x, y) = -\frac{7y}{\sqrt{20 - x^2 - 7y^2}}$, so $f_x(2, 1) = -\frac{2}{3}$ and $f_y(2, 1) = -\frac{7}{3}$. Then the linear approximation of f at $(2, 1)$ is given by
- $$\begin{aligned} f(x, y) &\approx f(2, 1) + f_x(2, 1)(x - 2) + f_y(2, 1)(y - 1) = 3 - \frac{2}{3}(x - 2) - \frac{7}{3}(y - 1) \\ &= -\frac{2}{3}x - \frac{7}{3}y + \frac{20}{3} \end{aligned}$$
- Thus $f(1.95, 1.08) \approx -\frac{2}{3}(1.95) - \frac{7}{3}(1.08) + \frac{20}{3} \approx 2.84\bar{6}$.

$$18. f(x, y) = \ln(x - 3y) \Rightarrow f_x(x, y) = \frac{1}{x - 3y} \text{ and } f_y(x, y) = -\frac{3}{x - 3y}, \text{ so } f_x(7, 2) = 1 \text{ and } f_y(7, 2) = -3.$$

Then the linear approximation of f at $(7, 2)$ is given by

$$\begin{aligned} f(x, y) &\approx f(7, 2) + f_x(7, 2)(x - 7) + f_y(7, 2)(y - 2) \\ &= 0 + 1(x - 7) - 3(y - 2) = x - 3y - 1 \end{aligned}$$

Thus $f(6.9, 2.06) \approx 6.9 - 3(2.06) - 1 = -0.28$. The graph shows that our approximated value is slightly greater than the actual value.



$$19. f(x, y, z) = \sqrt{x^2 + y^2 + z^2} \Rightarrow f_x(x, y, z) = \frac{x}{\sqrt{x^2 + y^2 + z^2}}, f_y(x, y, z) = \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \text{ and } f_z(x, y, z) = \frac{z}{\sqrt{x^2 + y^2 + z^2}}, \text{ so } f_x(3, 2, 6) = \frac{3}{7}, f_y(3, 2, 6) = \frac{2}{7}, \text{ and } f_z(3, 2, 6) = \frac{6}{7}. \text{ Then the linear approximation of } f \text{ at } (3, 2, 6) \text{ is given by}$$

$$\begin{aligned} f(x, y, z) &\approx f(3, 2, 6) + f_x(3, 2, 6)(x - 3) + f_y(3, 2, 6)(y - 2) + f_z(3, 2, 6)(z - 6) \\ &= 7 + \frac{3}{7}(x - 3) + \frac{2}{7}(y - 2) + \frac{6}{7}(z - 6) = \frac{3}{7}x + \frac{2}{7}y + \frac{6}{7}z \end{aligned}$$

$$\text{Thus } \sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2} = f(3.02, 1.97, 5.99) \approx \frac{3}{7}(3.02) + \frac{2}{7}(1.97) + \frac{6}{7}(5.99) \approx 6.9914.$$

20. From the table, $f(40, 20) = 28$. To estimate $f_v(40, 20)$ and $f_t(40, 20)$ we follow the procedure used in Exercise 15.3.4 [ET 14.3.4]. Since $f_v(40, 20) = \lim_{h \rightarrow 0} \frac{f(40 + h, 20) - f(40, 20)}{h}$, we approximate this quantity

$$\text{with } h = \pm 10 \text{ and use the values given in the table: } f_v(40, 20) \approx \frac{f(50, 20) - f(40, 20)}{10} = \frac{40 - 28}{10} = 1.2,$$

$$f_v(40, 20) \approx \frac{f(30, 20) - f(40, 20)}{-10} = \frac{17 - 28}{-10} = 1.1. \text{ Averaging these values gives } f_v(40, 20) \approx 1.15.$$

Similarly, $f_t(40, 20) = \lim_{h \rightarrow 0} \frac{f(40, 20 + h) - f(40, 20)}{h}$, so we use $h = 10$ and $h = -5$:

$$f_t(40, 20) \approx \frac{f(40, 30) - f(40, 20)}{10} = \frac{31 - 28}{10} = 0.3, f_t(40, 20) \approx \frac{f(40, 15) - f(40, 20)}{-5} = \frac{25 - 28}{-5} = 0.6.$$

Averaging these values gives $f_t(40, 15) \approx 0.45$. The linear approximation, then, is

$$\begin{aligned} f(v, t) &\approx f(40, 20) + f_v(40, 20)(v - 40) + f_t(40, 20)(t - 20) \\ &\approx 28 + 1.15(v - 40) + 0.45(t - 20) \end{aligned}$$

When $v = 43$ and $t = 24$, we estimate $f(43, 24) \approx 28 + 1.15(43 - 40) + 0.45(24 - 20) = 33.25$, so we would expect the wave heights to be approximately 33.25 ft.

21. From the table, $f(94, 80) = 127$. To estimate $f_T(94, 80)$ and $f_H(94, 80)$ we follow the procedure used in Section 15.3 [ET 14.3]. Since $f_T(94, 80) = \lim_{h \rightarrow 0} \frac{f(94 + h, 80) - f(94, 80)}{h}$, we approximate this quantity

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with $h = \pm 2$ and use the values given in the table: $f_T(94, 80) \approx \frac{f(96, 80) - f(94, 80)}{2} = \frac{135 - 127}{2} = 4$,

$$f_T(94, 80) \approx \frac{f(92, 80) - f(94, 80)}{-2} = \frac{119 - 127}{-2} = 4.$$

Averaging these values gives $f_T(94, 80) \approx 4$. Similarly, $f_H(94, 80) = \lim_{h \rightarrow 0} \frac{f(94, 80 + h) - f(94, 80)}{h}$,

$$\text{so we use } h = \pm 5: f_H(94, 80) \approx \frac{f(94, 85) - f(94, 80)}{5} = \frac{132 - 127}{5} = 1,$$

$f_H(94, 80) \approx \frac{f(94, 75) - f(94, 80)}{-5} = \frac{122 - 127}{-5} = 1$. Averaging these values gives $f_H(94, 80) \approx 1$. The linear approximation, then, is

$$\begin{aligned} f(T, H) &\approx f(94, 80) + f_T(94, 80)(T - 94) + f_H(94, 80)(H - 80) \\ &\approx 127 + 4(T - 94) + 1(H - 80) \end{aligned}$$

Thus when $T = 95$ and $H = 78$, $f(95, 78) \approx 127 + 4(95 - 94) + 1(78 - 80) = 129$, so we estimate the heat index to be approximately 129°F .

22. From the table, $f(-15, 50) = -29$. To estimate $f_T(-15, 50)$ and $f_v(-15, 50)$ we follow the procedure used in Section 15.3 [ET 14.3]. Since $f_T(-15, 50) = \lim_{h \rightarrow 0} \frac{f(-15 + h, 50) - f(-15, 50)}{h}$, we approximate this quantity with $h = \pm 5$ and use the values given in the table:

$$f_T(-15, 50) \approx \frac{f(-10, 50) - f(-15, 50)}{5} = \frac{-22 - (-29)}{5} = 1.4.$$

$$f_T(-15, 50) \approx \frac{f(-20, 50) - f(-15, 50)}{-5} = \frac{-35 - (-29)}{-5} = 1.2.$$

Averaging these values gives $f_T(-15, 50) \approx 1.3$. Similarly $f_v(-15, 50) = \lim_{h \rightarrow 0} \frac{f(-15, 50 + h) - f(-15, 50)}{h}$ so

$$\text{we use } h = \pm 10: f_v(-15, 50) \approx \frac{f(-15, 60) - f(-15, 50)}{10} = \frac{-30 - (-29)}{10} = -0.1,$$

$$f_v(-15, 50) \approx \frac{f(-15, 40) - f(-15, 50)}{-10} = \frac{-27 - (-29)}{-10} = -0.2.$$

Averaging these values gives $f_v(-15, 50) \approx -0.15$. The linear approximation to the wind-chill index function, then, is

$$\begin{aligned} f(T, v) &\approx f(-15, 50) + f_T(-15, 50)(T - (-15)) + f_v(-15, 50)(v - 50) \\ &\approx -29 + (1.3)(T + 15) - (0.15)(v - 50) \end{aligned}$$

Thus when $T = -17^\circ\text{C}$ and $v = 55 \text{ km/h}$, $f(-17, 55) \approx -29 + (1.3)(-17 + 15) - (0.15)(55 - 50) = -32.35$, so we estimate the wind-chill index to be approximately -32.35°C .

23. $z = x^3 \ln(y^2) \Rightarrow$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = 3x^2 \ln(y^2) dx + x^3 \cdot \frac{1}{y^2} (2y) dy = 3x^2 \ln(y^2) dx + \frac{2x^3}{y} dy.$$

24. $v = y \cos xy \Rightarrow$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = y(-\sin xy)y dx + [y(-\sin xy)x + \cos xy] dy = -y^2 \sin xy dx + (\cos xy - xy \sin xy) dy$$

25. $u = e^t \sin \theta \Rightarrow du = \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial \theta} d\theta = e^t \sin \theta dt + e^t \cos \theta d\theta$

26. $u = \frac{r}{s+2t} \Rightarrow$

$$\begin{aligned} du &= \frac{\partial u}{\partial r} dr + \frac{\partial u}{\partial s} ds + \frac{\partial u}{\partial t} dt = \frac{1}{s+2t} dr + r(-1)(s+2t)^{-2} ds + r(-1)(s+2t)^{-2}(2) dt \\ &= \frac{1}{s+2t} dr - \frac{r}{(s+2t)^2} ds - \frac{2r}{(s+2t)^2} dt \end{aligned}$$

27. $w = \ln \sqrt{x^2 + y^2 + z^2} \Rightarrow$

$$\begin{aligned} dw &= \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz \\ &= \left(\frac{1}{2} \right) \frac{2x(x^2 + y^2 + z^2)^{-1/2} dx + 2y(x^2 + y^2 + z^2)^{-1/2} dy + 2z(x^2 + y^2 + z^2)^{-1/2} dz}{(x^2 + y^2 + z^2)^{1/2}} \\ &= \frac{x dx + y dy + z dz}{x^2 + y^2 + z^2} \end{aligned}$$

28. $w = xye^{xz} \Rightarrow$

$$\begin{aligned} dw &= \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz = (xyze^{xz} + ye^{xz}) dx + xe^{xz} dy + x^2 ye^{xz} dz \\ &= (xz + 1)ye^{xz} dx + xe^{xz} dy + x^2 ye^{xz} dz. \end{aligned}$$

29. $dx = \Delta x = 0.05$, $dy = \Delta y = 0.1$, $z = 5x^2 + y^2$, $z_x = 10x$, $z_y = 2y$. Thus when $x = 1$ and $y = 2$, $dz = z_x(1, 2) dx + z_y(1, 2) dy = (10)(0.05) + (4)(0.1) = 0.9$ while $\Delta z = f(1.05, 2.1) - f(1, 2) = 5(1.05)^2 + (2.1)^2 - 5 - 4 = 0.9225$.

30. $dx = \Delta x = -0.04$, $dy = \Delta y = 0.05$, $z = x^2 - xy + 3y^2$, $z_x = 2x - y$, $z_y = 6y - x$. Thus when $x = 3$ and $y = -1$, $dz = (7)(-0.04) + (-9)(0.05) = -0.73$ while $\Delta z = (2.96)^2 - (2.96)(-0.95) + 3(-0.95)^2 - (9 + 3 + 3) = -0.7189$.

31. $dA = \frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy = y dx + x dy$ and $|\Delta x| \leq 0.1$, $|\Delta y| \leq 0.1$. We use $dx = 0.1$, $dy = 0.1$ with $x = 30$, $y = 24$; then the maximum error in the area is about $dA = 24(0.1) + 30(0.1) = 5.4 \text{ cm}^2$.

32. Let S be surface area. Then $S = 2(xy + xz + yz)$ and $dS = 2(y + z) dx + 2(x + z) dy + 2(x + y) dz$. The maximum error occurs with $\Delta x = \Delta y = \Delta z = 0.2$. Using $dx = \Delta x$, $dy = \Delta y$, $dz = \Delta z$ we find the maximum error in calculated surface area to be about $dS = (220)(0.2) + (260)(0.2) + (280)(0.2) = 152 \text{ cm}^2$.

33. The volume of a can is $V = \pi r^2 h$ and $\Delta V \approx dV$ is an estimate of the amount of tin. Here $dV = 2\pi r h dr + \pi r^2 dh$, so put $dr = 0.04$, $dh = 0.08$ (0.04 on top, 0.04 on bottom) and then $\Delta V \approx dV = 2\pi(48)(0.04) + \pi(16)(0.08) \approx 16.08 \text{ cm}^3$. Thus the amount of tin is about 16 cm^3 .

34. Let V be the volume. Then $V = \pi r^2 h$ and $\Delta V \approx dV = 2\pi r h dr + \pi r^2 dh$ is an estimate of the amount of metal. With $dr = 0.05$ and $dh = 0.2$ we get $dV = 2\pi(2)(10)(0.05) + \pi(2)^2(0.2) = 2.80\pi \approx 8.8 \text{ cm}^3$.

35. The area of the rectangle is $A = xy$, and $\Delta A \approx dA$ is an estimate of the area of paint in the stripe. Here $dA = y dx + x dy$, so with $dx = dy = \frac{3-3}{12} = \frac{1}{2}$, $\Delta A \approx dA = (100)\left(\frac{1}{2}\right) + (200)\left(\frac{1}{2}\right) = 150 \text{ ft}^2$. Thus there are approximately 150 ft^2 of paint in the stripe.

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36. Here $dV = \Delta V = 0.3$, $dT = \Delta T = -5$, $P = 8.31 \frac{T}{V}$, so

$$dP = \left(\frac{8.31}{V} \right) dT - \frac{8.31 \cdot T}{V^2} dV = 8.31 \left[-\frac{5}{12} - \frac{310}{144} \cdot \frac{3}{10} \right] \approx -8.83.$$

Thus the pressure will drop by about 8.83 kPa.

37. First we find $\frac{\partial R}{\partial R_1}$ implicitly by taking partial derivatives of both sides with respect to R_1 :

$$\frac{\partial}{\partial R_1} \left[\frac{1}{R} \right] = \frac{\partial [(1/R_1) + (1/R_2) + (1/R_3)]}{\partial R_1} \Rightarrow -R^{-2} \frac{\partial R}{\partial R_1} = -R_1^{-2} \Rightarrow \frac{\partial R}{\partial R_1} = \frac{R^2}{R_1^2}. \text{ Then by}$$

symmetry, $\frac{\partial R}{\partial R_2} = \frac{R^2}{R_2^2}$, $\frac{\partial R}{\partial R_3} = \frac{R^2}{R_3^2}$. When $R_1 = 25$, $R_2 = 40$ and $R_3 = 50$, $\frac{1}{R} = \frac{17}{200} \Leftrightarrow R = \frac{200}{17}$ ohms.

Since the possible error for each R_i is 0.5%, the maximum error of R is attained by setting $\Delta R_i = 0.005R_i$. So

$$\begin{aligned} \Delta R \approx dR &= \frac{\partial R}{\partial R_1} \Delta R_1 + \frac{\partial R}{\partial R_2} \Delta R_2 + \frac{\partial R}{\partial R_3} \Delta R_3 = (0.005)R^2 \left[\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right] \\ &= (0.005)R = \frac{1}{17} \approx 0.059 \text{ ohms} \end{aligned}$$

38. Let x, y, z and w be the four numbers with $p(x, y, z, w) = xyzw$. Since the largest error due to rounding for each number is 0.05, the maximum error in the calculated product is approximated by

$dp = (yzw)(0.05) + (xzw)(0.05) + (xyw)(0.05) + (xyz)(0.05)$. Furthermore, each of the numbers is positive but less than 50, so the product of any three is between 0 and $(50)^3$. Thus $dp \leq 4(50)^3(0.05) = 25,000$.

39. $\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b) = (a + \Delta x)^2 + (b + \Delta y)^2 - (a^2 + b^2)$

$$= a^2 + 2a \Delta x + (\Delta x)^2 + b^2 + 2b \Delta y + (\Delta y)^2 - a^2 - b^2 = 2a \Delta x + (\Delta x)^2 + 2b \Delta y + (\Delta y)^2$$

But $f_x(a, b) = 2a$ and $f_y(a, b) = 2b$ and so $\Delta z = f_x(a, b) \Delta x + f_y(a, b) \Delta y + \Delta x \Delta x + \Delta y \Delta y$, which is Definition 7 with $\varepsilon_1 = \Delta x$ and $\varepsilon_2 = \Delta y$. Hence f is differentiable.

40. $\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b) = (a + \Delta x)(b + \Delta y) - 5(b + \Delta y)^2 - (ab - 5b^2)$

$$= ab + a \Delta y + b \Delta x + \Delta x \Delta y - 5b^2 - 10b \Delta y - 5(\Delta y)^2 - ab + 5b^2$$

$$= (a - 10b) \Delta y + b \Delta x + \Delta x \Delta y - 5 \Delta y \Delta y,$$

but $f_x(a, b) = b$ and $f_y(a, b) = a - 10b$ and so $\Delta z = f_x(a, b) \Delta x + f_y(a, b) \Delta y + \Delta x \Delta y - 5 \Delta y \Delta y$, which is Definition 7 with $\varepsilon_1 = \Delta y$ and $\varepsilon_2 = -5 \Delta y$. Hence f is differentiable.

41. To show that f is continuous at (a, b) we need to show that $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = f(a, b)$ or equivalently

$$\lim_{(\Delta x, \Delta y) \rightarrow (0,0)} f(a + \Delta x, b + \Delta y) = f(a, b). \text{ Since } f \text{ is differentiable at } (a, b),$$

$f(a + \Delta x, b + \Delta y) - f(a, b) = \Delta z = f_x(a, b) \Delta x + f_y(a, b) \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$, where ε_1 and $\varepsilon_2 \rightarrow 0$ as $(\Delta x, \Delta y) \rightarrow (0, 0)$. Thus $f(a + \Delta x, b + \Delta y) = f(a, b) + f_x(a, b) \Delta x + f_y(a, b) \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$. Taking the limit of both sides as $(\Delta x, \Delta y) \rightarrow (0, 0)$ gives $\lim_{(\Delta x, \Delta y) \rightarrow (0,0)} f(a + \Delta x, b + \Delta y) = f(a, b)$. Thus f is continuous at (a, b) .

42. (a) $\lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{0 - 0}{h} = 0$ and $\lim_{h \rightarrow 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{0 - 0}{h} = 0$. Thus

$f_x(0, 0) = f_y(0, 0) = 0$. To show that f isn't differentiable at $(0, 0)$ we need only show that f is not continuous at $(0, 0)$ and apply Exercise 41. As $(x, y) \rightarrow (0, 0)$ along the x -axis $f(x, y) = 0/x^2 = 0$ for $x \neq 0$ so

$f(x, y) \rightarrow 0$ as $(x, y) \rightarrow (0, 0)$ along the x -axis. But as $(x, y) \rightarrow (0, 0)$ along the line $y = x$,
 $f(x, x) = x^2/(2x^2) = \frac{1}{2}$ for $x \neq 0$ so $f(x, y) \rightarrow \frac{1}{2}$ as $(x, y) \rightarrow (0, 0)$ along this line. Thus $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$
 doesn't exist, so f is discontinuous at $(0, 0)$ and thus not differentiable there.

- (b) For $(x, y) \neq (0, 0)$, $f_x(x, y) = \frac{(x^2 + y^2)y - xy(2x)}{(x^2 + y^2)^2} = \frac{y(y^2 - x^2)}{(x^2 + y^2)^2}$. If we approach $(0, 0)$ along the y -axis,
 then $f_x(x, y) = f_x(0, y) = \frac{y^3}{y^4} = \frac{1}{y}$, so $f_x(x, y) \rightarrow \pm\infty$ as $(x, y) \rightarrow (0, 0)$. Thus $\lim_{(x,y) \rightarrow (0,0)} f_x(x, y)$ does
 not exist and $f_x(x, y)$ is not continuous at $(0, 0)$. Similarly, $f_y(x, y) = \frac{(x^2 + y^2)x - xy(2y)}{(x^2 + y^2)^2} = \frac{x(x^2 - y^2)}{(x^2 + y^2)^2}$
 for $(x, y) \neq (0, 0)$, and if we approach $(0, 0)$ along the x -axis, then $f_y(x, y) = f_y(x, 0) = \frac{x^3}{x^4} = \frac{1}{x}$. Thus
 $\lim_{(x,y) \rightarrow (0,0)} f_y(x, y)$ does not exist and $f_y(x, y)$ is not continuous at $(0, 0)$.

15.5 The Chain Rule

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- $z = x^2y + xy^3$, $x = 2 + t^4$, $y = 1 - t^3 \Rightarrow$
 $\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = (2xy + y^2)(4t^3) + (x^2 + 2xy)(-3t^2) = 4(2xy + y^2)t^3 - 3(x^2 + 2xy)t^2$
- $z = \sqrt{x^2 + y^2}$, $x = e^{2t}$, $y = e^{-2t} \Rightarrow$
 $\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = \frac{1}{2}(x^2 + y^2)^{-1/2}(2x) \cdot e^{2t}(2) + \frac{1}{2}(x^2 + y^2)^{-1/2}(2y) \cdot e^{-2t}(-2) = \frac{2xe^{2t} - 2ye^{2t}}{\sqrt{x^2 + y^2}}$
- $z = \sin x \cos y$, $x = \pi t$, $y = \sqrt{t} \Rightarrow$
 $\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = \cos x \cos y \cdot \pi + \sin x (-\sin y) \cdot \frac{1}{2}t^{-1/2} = \pi \cos x \cos y - \frac{1}{2\sqrt{t}} \sin x \sin y$
- $z = x \ln(x + 2y)$, $x = \sin t$, $y = \cos t \Rightarrow$
 $\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = \left[x \cdot \frac{1}{x + 2y} + 1 \cdot \ln(x + 2y) \right] \cos t + x \cdot \frac{1}{x + 2y} (2) \cdot (-\sin t)$
 $= \left[\frac{x}{x + 2y} + \ln(x + 2y) \right] \cos t - \frac{2x}{x + 2y} (\sin t)$
- $w = xe^{y/z}$, $x = t^3$, $y = 1 - t$, $z = 1 + 2t \Rightarrow$
 $\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} = e^{y/z} \cdot 2t + xe^{y/z} \left(\frac{1}{z} \right) \cdot (-1) + xe^{y/z} \left(-\frac{y}{z^2} \right) \cdot 2 = e^{y/z} \left(2t - \frac{x}{z} - \frac{2xy}{z^2} \right)$
- $w = xy + yz^2$, $x = e^t$, $y = e^t \sin t$, $z = e^t \cos t \Rightarrow$
 $\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} = y \cdot e^t + (x + z^2) \cdot (e^t \cos t + e^t \sin t) + 2yz \cdot (-e^t \sin t + e^t \cos t)$
 $= e^t [y + (x + z^2)(\cos t + \sin t) + 2yz(\cos t - \sin t)]$
- $z = x^2 + xy + y^2$, $x = s + t$, $y = st \Rightarrow$
 $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = (2x + y)(1) + (x + 2y)(t) = 2x + y + xt + 2yt$
 $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = (2x + y)(1) + (x + 2y)(s) = 2x + y + xs + 2ys$