**42.** Let $\epsilon > 0$ be given. We need to find $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{a})| < \epsilon$ whenever $|\mathbf{x} - \mathbf{a}| < \delta$ or $|\mathbf{c} \cdot \mathbf{x} - \mathbf{c} \cdot \mathbf{a}| < \epsilon$ whenever $|\mathbf{x} - \mathbf{a}| < \delta$. But $|\mathbf{c} \cdot \mathbf{x} - \mathbf{c} \cdot \mathbf{a}| = |\mathbf{c} \cdot (\mathbf{x} - \mathbf{a})|$ and $|\mathbf{c} \cdot (\mathbf{x} - \mathbf{a})| \leq |\mathbf{c}|\,|\mathbf{x} - \mathbf{a}|$ by Exercise 13.3.57 [ET 12.3.57] (the Cauchy-Schwarz Inequality). Let $\epsilon > 0$ be given and set $\delta = \epsilon/|\mathbf{c}|$. Then whenever $0 < |\mathbf{x} - \mathbf{a}| < \delta$, $|f(\mathbf{x}) - f(\mathbf{a})| = |\mathbf{c} \cdot \mathbf{x} - \mathbf{c} \cdot \mathbf{a}| \leq |\mathbf{c}|\,|\mathbf{x} - \mathbf{a}| < |\mathbf{c}|\,\delta = |\mathbf{c}|\,(\epsilon/|\mathbf{c}|) = \epsilon$. So $f$ is continuous on $\mathbb{R}^n$.

## 15.3 Partial Derivatives ET 14.3

**1.** (a) $\partial T/\partial x$ represents the rate of change of $T$ when we fix $y$ and $t$ and consider $T$ as a function of the single variable $x$, which describes how quickly the temperature changes when longitude changes but latitude and time are constant. $\partial T/\partial y$ represents the rate of change of $T$ when we fix $x$ and $t$ and consider $T$ as a function of $y$, which describes how quickly the temperature changes when latitude changes but longitude and time are constant. $\partial T/\partial t$ represents the rate of change of $T$ when we fix $x$ and $y$ and consider $T$ as a function of $t$, which describes how quickly the temperature changes over time for a constant longitude and latitude.

(b) $f_x(158, 21, 9)$ represents the rate of change of temperature at longitude 158°W, latitude 21°N at 9:00 A.M. when only longitude varies. Since the air is warmer to the west than to the east, increasing longitude results in an increased air temperature, so we would expect $f_x(158, 21, 9)$ to be positive. $f_y(158, 21, 9)$ represents the rate of change of temperature at the same time and location when only latitude varies. Since the air is warmer to the south and cooler to the north, increasing latitude results in a decreased air temperature, so we would expect $f_y(158, 21, 9)$ to be negative. $f_t(158, 21, 9)$ represents the rate of change of temperature at the same time and location when only time varies. Since typically air temperature increases from the morning to the afternoon as the sun warms it, we would expect $f_t(158, 21, 9)$ to be positive.

**2.** By Definition 4, $f_T(92, 60) = \lim\limits_{h \to 0} \dfrac{f(92 + h, 60) - f(92, 60)}{h}$, which we can approximate by considering $h = 2$ and $h = -2$ and using the values given in Table 1: $f_T(92, 60) \approx \dfrac{f(94, 60) - f(92, 60)}{2} = \dfrac{111 - 105}{2} = 3$, $f_T(92, 60) \approx \dfrac{f(90, 60) - f(92, 60)}{-2} = \dfrac{100 - 105}{-2} = 2.5$. Averaging these values, we estimate $f_T(92, 60)$ to be approximately 2.75. Thus, when the actual temperature is 92°F and the relative humidity is 60%, the apparent temperature rises by about 2.75°F for every degree that the actual temperature rises.

Similarly, $f_H(92, 60) = \lim\limits_{h \to 0} \dfrac{f(92, 60 + h) - f(92, 60)}{h}$ which we can approximate by considering $h = 5$ and $h = -5$: $f_H(92, 60) \approx \dfrac{f(92, 65) - f(92, 60)}{5} = \dfrac{108 - 105}{5} = 0.6$, $f_H(92, 60) \approx \dfrac{f(92, 55) - f(92, 60)}{-5} = \dfrac{103 - 105}{-5} = 0.4$. Averaging these values, we estimate $f_H(92, 60)$ to be approximately 0.5. Thus, when the actual temperature is 92°F and the relative humidity is 60%, the apparent temperature rises by about 0.5°F for every percent that the relative humidity increases.

**3.** (a) By Definition 4, $f_T(-15,30) = \lim_{h\to 0} \frac{f(-15+h,30) - f(-15,30)}{h}$, which we can approximate by considering $h = 5$ and $h = -5$ and using the values given in the table:

$f_T(-15,30) \approx \frac{f(-10,30) - f(-15,30)}{5} = \frac{-20-(-26)}{5} = \frac{6}{5} = 1.2$,

$f_T(-15,30) \approx \frac{f(-20,30) - f(-15,30)}{-5} = \frac{-33-(-26)}{-5} = \frac{-7}{-5} = 1.4$. Averaging these values, we estimate $f_T(-15,30)$ to be approximately 1.3. Thus, when the actual temperature is $-15°C$ and the wind speed is 30 km/h, the apparent temperature rises by about 1.3°C for every degree that the actual temperature rises.

Similarly, $f_v(-15,30) = \lim_{h\to 0} \frac{f(-15,30+h) - f(-15,30)}{h}$ which we can approximate by considering $h = 10$ and $h = -10$: $f_v(-15,30) \approx \frac{f(-15,40) - f(-15,30)}{10} = \frac{-27-(-26)}{10} = \frac{-1}{10} = -0.1$,

$f_v(-15,30) \approx \frac{f(-15,20) - f(-15,30)}{-10} = \frac{-24-(-26)}{-10} = \frac{2}{-10} = -0.2$. Averaging these values, we estimate $f_v(-15,30)$ to be approximately $-0.15$. Thus, when the actual temperature is $-15°C$ and the wind speed is 30 km/h, the apparent temperature decreases by about 0.15°C for every km/h that the wind speed increases.

(b) For a fixed wind speed $v$, the values of the wind-chill index $W$ increase as temperature $T$ increases (look at a column of the table), so $\frac{\partial W}{\partial T}$ is positive. For a fixed temperature $T$, the values of $W$ decrease (or remain constant) as $v$ increases (look at a row of the table), so $\frac{\partial W}{\partial v}$ is negative (or perhaps 0).

(c) For fixed values of $T$, the function values $f(T,v)$ appear to become constant (or nearly constant) as $v$ increases, so the corresponding rate of change is 0 or near 0 as $v$ increases. This suggests that $\lim_{v\to\infty} (\partial W/\partial v) = 0$.

**4.** (a) $\partial h/\partial v$ represents the rate of change of $h$ when we fix $t$ and consider $h$ as a function of $v$, which describes how quickly the wave heights change when the wind speed changes for a fixed time duration. $\partial h/\partial t$ represents the rate of change of $h$ when we fix $v$ and consider $h$ as a function of $t$, which describes how quickly the wave heights change when the duration of time changes, but the wind speed is constant.

(b) By Definition 4, $f_v(40,15) = \lim_{h\to 0} \frac{f(40+h,15) - f(40,15)}{h}$ which we can approximate by considering $h = 10$ and $h = -10$ and using the values given in the table:

$f_v(40,15) \approx \frac{f(50,15) - f(40,15)}{10} = \frac{36-25}{10} = 1.1$,

$f_v(40,15) \approx \frac{f(30,15) - f(40,15)}{-10} = \frac{16-25}{-10} = 0.9$. Averaging these values, we have $f_v(40,15) \approx 1.0$. Thus, when a 40-knot wind has been blowing for 15 hours, the wave heights should increase by about 1 foot for every knot that the wind speed increases (with the same time duration). Similarly,

$f_t(40,15) = \lim_{h\to 0} \frac{f(40,15+h) - f(40,15)}{h}$ which we can approximate by considering

$h = 5$ and $h = -5$: $f_t(40,15) \approx \frac{f(40,20) - f(40,15)}{5} = \frac{28-25}{5} = 0.6$,

$f_t(40,15) \approx \dfrac{f(40,10) - f(40,15)}{-5} = \dfrac{21-25}{-5} = 0.8$. Averaging these values, we have $f_t(40,15) \approx 0.7$. Thus, when a 40-knot wind has been blowing for 15 hours, the wave heights increase by about 0.7 feet for every additional hour that the wind blows.

(c) For fixed values of $v$, the function values $f(v,t)$ appear to increase in smaller and smaller increments, becoming nearly constant as $t$ increases. Thus, the corresponding rate of change is nearly 0 as $t$ increases, suggesting that $\lim\limits_{t\to\infty} (\partial h/\partial t) = 0$.

**5.** (a) If we start at $(1,2)$ and move in the positive $x$-direction, the graph of $f$ increases. Thus $f_x(1,2)$ is positive.

(b) If we start at $(1,2)$ and move in the positive $y$-direction, the graph of $f$ decreases. Thus $f_y(1,2)$ is negative.

**6.** (a) The graph of $f$ decreases if we start at $(-1,2)$ and move in the positive $x$-direction, so $f_x(-1,2)$ is negative.

(b) The graph of $f$ decreases if we start at $(-1,2)$ and move in the positive $y$-direction, so $f_y(-1,2)$ is negative.

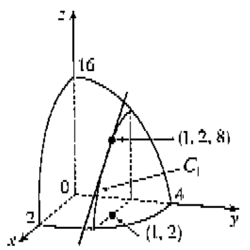
(c) $f_{xx} = \frac{\partial}{\partial x}(f_x)$, so $f_{xx}$ is the rate of change of $f_x$ in the $x$-direction. $f_x$ is negative at $(-1,2)$ and if we move in the positive $x$-direction, the surface becomes less steep. Thus the values of $f_x$ are increasing and $f_{xx}(-1,2)$ is positive.

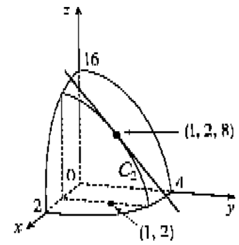
(d) $f_{yy}$ is the rate of change of $f_y$ in the $y$-direction. $f_y$ is negative at $(-1,2)$ and if we move in the positive $y$-direction, the surface becomes steeper. Thus the values of $f_y$ are decreasing, and $f_{yy}(-1,2)$ is negative.

**7.** First of all, if we start at the point $(3,-3)$ and move in the positive $y$-direction, we see that both $b$ and $c$ decrease, while $a$ increases. Both $b$ and $c$ have a low point at about $(3,-1.5)$, while $a$ is 0 at this point. So $a$ is definitely the graph of $f_y$, and one of $b$ and $c$ is the graph of $f$. To see which is which, we start at the point $(-3,-1.5)$ and move in the positive $x$-direction. $b$ traces out a line with negative slope, while $c$ traces out a parabola opening downward. This tells us that $b$ is the $x$-derivative of $c$. So $c$ is the graph of $f$, $b$ is the graph of $f_x$, and $a$ is the graph of $f_y$.

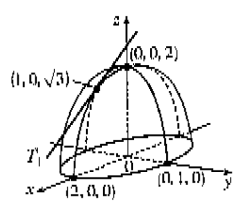
**8.** $f_x(2,1)$ is the rate of change of $f$ at $(2,1)$ in the $x$-direction. If we start at $(2,1)$, where $f(2,1) = 10$, and move in the positive $x$-direction, we reach the next contour line (where $f(x,y) = 12$) after approximately 0.6 units. This represents an average rate of change of about $\frac{2}{0.6}$. If we approach the point $(2,1)$ from the left (moving in the positive $x$-direction) the output values increase from 8 to 10 with an increase in $x$ of approximately 0.9 units, corresponding to an average rate of change of $\frac{2}{0.9}$. A good estimate for $f_x(2,1)$ would be the average of these two, so $f_x(2,1) \approx 2.8$. Similarly, $f_y(2,1)$ is the rate of change of $f$ at $(2,1)$ in the $y$-direction. If we approach $(2,1)$ from below, the output values decrease from 12 to 10 with a change in $y$ of approximately 1 unit, corresponding to an average rate of change of $-2$. If we start at $(2,1)$ and move in the positive $y$-direction, the output values decrease from 10 to 8 after approximately 0.9 units, a rate of change of $\frac{-2}{0.9}$. Averaging these two results, we estimate $f_y(2,1) \approx -2.1$.

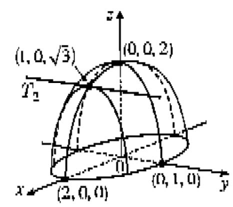
**9.** $f(x, y) = 16 - 4x^2 - y^2 \Rightarrow f_x(x, y) = -8x$ and $f_y(x, y) = -2y \Rightarrow f_x(1, 2) = -8$ and $f_y(1, 2) = -4$.
The graph of $f$ is the paraboloid $z = 16 - 4x^2 - y^2$ and the vertical plane $y = 2$ intersects it in the parabola $z = 12 - 4x^2$, $y = 2$ (the curve $C_1$ in the first figure).
The slope of the tangent line to this parabola at $(1, 2, 8)$ is $f_x(1, 2) = -8$. Similarly the plane $x = 1$ intersects the paraboloid in the parabola $z = 12 - y^2$, $x = 1$ (the curve $C_2$ in the second figure) and the slope of the tangent line at $(1, 2, 8)$ is $f_y(1, 2) = -4$.



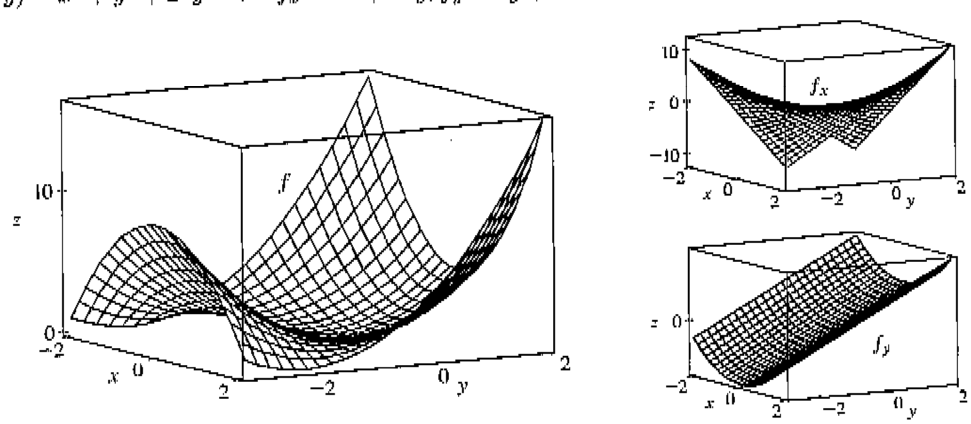




**10.** $f(x, y) = (4 - x^2 - 4y^2)^{1/2} \Rightarrow f_x(x, y) = -x(4 - x^2 - 4y^2)^{-1/2}$ and $f_y(x, y) = -4y(4 - x^2 - 4y^2)^{-1/2}$
$\Rightarrow f_x(1, 0) = -\frac{1}{\sqrt{3}}$, $f_y(1, 0) = 0$. The graph of $f$ is the upper half of the ellipsoid $z^2 + x^2 + 4y^2 = 4$ and the plane $y = 0$ intersects the graph in the semicircle $x^2 + z^2 = 4$, $z \geq 0$ and the slope of the tangent line $T_1$ to this semicircle at $(1, 0, \sqrt{3})$ is $f_x(1, 0) = -\frac{1}{\sqrt{3}}$. Similarly the plane $x = 1$ intersects the graph in the semi-ellipse $z^2 + 4y^2 = 3$, $z \geq 0$ and the slope of the tangent line $T_2$ to this semi-ellipse at $(1, 0, \sqrt{3})$ is $f_y(1, 0) = 0$.



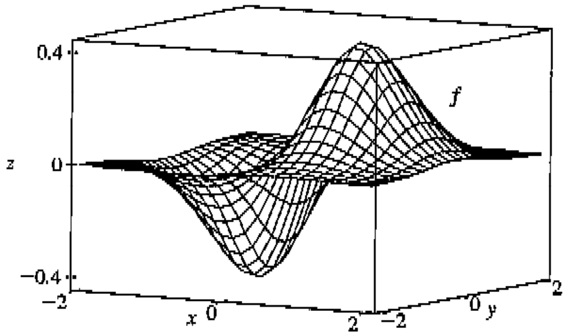



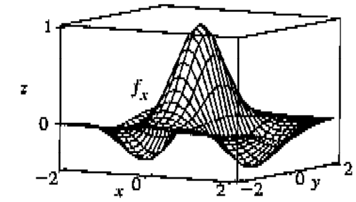

**11.** $f(x, y) = x^2 + y^2 + x^2 y \Rightarrow f_x = 2x + 2xy$, $f_y = 2y + x^2$

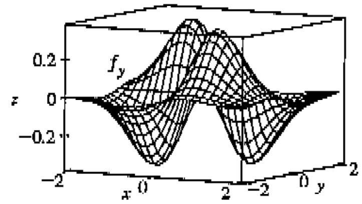



Note that the traces of $f$ in planes parallel to the $xz$-plane are parabolas which open downward for $y < -1$ and upward for $y > -1$, and the traces of $f_x$ in these planes are straight lines, which have negative slopes for $y < -1$ and positive slopes for $y > -1$. The traces of $f$ in planes parallel to the $yz$-plane are parabolas which always open upward, and the traces of $f_y$ in these planes are straight lines with positive slopes.

**12.** $f(x,y) = xe^{-x^2-y^2} \;\Rightarrow\; f_x = x\left(-2xe^{-x^2-y^2}\right) + e^{-x^2-y^2} = e^{-x^2-y^2}(1-2x^2)$, $f_y = -2xye^{-x^2-y^2}$




Note that traces of $f$ in planes parallel to the $xz$-plane have two extreme values, while traces of $f_x$ in these planes have two zeros. Traces of $f$ in planes parallel to the $yz$-plane have only one extreme value (a minimum if $x < 0$, a maximum if $x > 0$), and traces of $f_y$ in these planes have only one zero (going from negative to positive if $x < 0$ and from positive to negative if $x > 0$).

**13.** $f(x,y) = 3x - 2y^4 \;\Rightarrow\; f_x(x,y) = 3 - 0 = 3$, $f_y(x,y) = 0 - 8y^3 = -8y^3$

**14.** $f(x,y) = x^5 + 3x^3y^2 + 3xy^4 \;\Rightarrow\; f_x(x,y) = 5x^4 + 3\cdot 3x^2\cdot y^2 + 3\cdot 1\cdot y^4 = 5x^4 + 9x^2y^2 + 3y^4$,
$f_y(x,y) = 0 + 3x^3\cdot 2y + 3x\cdot 4y^3 = 6x^3y + 12xy^3$.

**15.** $z = xe^{3y} \;\Rightarrow\; \dfrac{\partial z}{\partial x} = e^{3y}$, $\dfrac{\partial z}{\partial y} = 3xe^{3y}$

**16.** $z = y\ln x \;\Rightarrow\; \dfrac{\partial z}{\partial x} = \dfrac{y}{x}$, $\dfrac{\partial z}{\partial y} = \ln x$

**17.** $f(x,y) = \dfrac{x-y}{x+y} \;\Rightarrow\; f_x(x,y) = \dfrac{(1)(x+y)-(x-y)(1)}{(x+y)^2} = \dfrac{2y}{(x+y)^2}$,
$f_y(x,y) = \dfrac{(-1)(x+y)-(x-y)(1)}{(x+y)^2} = -\dfrac{2x}{(x+y)^2}$

**18.** $f(x,y) = x^y \;\Rightarrow\; f_x(x,y) = yx^{y-1}$, $f_y(x,y) = x^y\ln x$

**19.** $w = \sin\alpha\cos\beta \;\Rightarrow\; \dfrac{\partial w}{\partial \alpha} = \cos\alpha\cos\beta$, $\dfrac{\partial w}{\partial \beta} = -\sin\alpha\sin\beta$

**20.** $f(s,t) = \dfrac{st^2}{s^2+t^2} \;\Rightarrow$
$f_s(s,t) = \dfrac{t^2(s^2+t^2) - st^2(2s)}{(s^2+t^2)^2} = \dfrac{t^4 - s^2t^2}{(s^2+t^2)^2}$, $f_t(s,t) = \dfrac{2st(s^2+t^2) - st^2(2t)}{(s^2+t^2)^2} = \dfrac{2s^3t}{(s^2+t^2)^2}$

**21.** $f(r,s) = r\ln(r^2+s^2) \;\Rightarrow\; f_r(r,s) = r\cdot\dfrac{2r}{r^2+s^2} + \ln(r^2+s^2)\cdot 1 = \dfrac{2r^2}{r^2+s^2} + \ln(r^2+s^2)$,

$f_s(r,s) = r\cdot\dfrac{2s}{r^2+s^2} + 0 = \dfrac{2rs}{r^2+s^2}$

**22.** $f(x,t) = \arctan(x\sqrt{t}) \;\Rightarrow\; f_x(x,t) = \dfrac{1}{1+(x\sqrt{t})^2}\cdot\sqrt{t} = \dfrac{\sqrt{t}}{1+x^2t}$,

$f_t(x,t) = \dfrac{1}{1+(x\sqrt{t})^2}\cdot x\left(\dfrac{1}{2}t^{-1/2}\right) = \dfrac{x}{2\sqrt{t}\,(1+x^2t)}$

**23.** $u = te^{w/t} \;\Rightarrow\; \dfrac{\partial u}{\partial t} = t\cdot e^{w/t}(-wt^{-2}) + e^{w/t}\cdot 1 = e^{w/t} - \dfrac{w}{t}e^{w/t} = e^{w/t}\left(1-\dfrac{w}{t}\right)$, $\dfrac{\partial u}{\partial w} = te^{w/t}\cdot\dfrac{1}{t} = e^{w/t}$

**24.** $f(x,y) = \displaystyle\int_y^x \cos(t^2)\,dt \;\Rightarrow\; f_x(x,y) = \dfrac{\partial}{\partial x}\int_y^x \cos(t^2)\,dt = \cos(x^2)$ by the Fundamental Theorem of Calculus, Part 1; $f_y(x,y) = \dfrac{\partial}{\partial y}\displaystyle\int_y^x \cos(t^2)\,dt = -\dfrac{\partial}{\partial y}\cos(t^2)\,dt = -\cos(y^2)$.

**25.** $f(x,y,z) = xy^2z^3 + 3yz \;\Rightarrow\; f_x(x,y,z) = y^2z^3$, $f_y(x,y,z) = 2xyz^3 + 3z$, $f_z(x,y,z) = 3xy^2z^2 + 3y$

**26.** $f(x,y,z) = x^2e^{yz} \;\Rightarrow\; f_x(x,y,z) = 2xe^{yz}$, $f_y(x,y,z) = x^2e^{yz}(z) = x^2ze^{yz}$,
$f_z(x,y,z) = x^2e^{yz}(y) = x^2ye^{yz}$

**27.** $w = \ln(x+2y+3z) \;\Rightarrow\; \dfrac{\partial w}{\partial x} = \dfrac{1}{x+2y+3z}$, $\dfrac{\partial w}{\partial y} = \dfrac{2}{x+2y+3z}$, $\dfrac{\partial w}{\partial z} = \dfrac{3}{x+2y+3z}$

**28.** $w = \sqrt{r^2+s^2+t^2} \;\Rightarrow\; \dfrac{\partial w}{\partial r} = \dfrac{1}{2}(r^2+s^2+t^2)^{-1/2}(2r) = \dfrac{r}{\sqrt{r^2+s^2+t^2}}$, $\dfrac{\partial w}{\partial s} = \dfrac{s}{\sqrt{r^2+s^2+t^2}}$,

$\dfrac{\partial w}{\partial t} = \dfrac{t}{\sqrt{r^2+s^2+t^2}}$

**29.** $u = xe^{-t}\sin\theta \;\Rightarrow\; \dfrac{\partial u}{\partial x} = e^{-t}\sin\theta$, $\dfrac{\partial u}{\partial t} = -xe^{-t}\sin\theta$, $\dfrac{\partial u}{\partial \theta} = xe^{-t}\cos\theta$

**30.** $u = x^{y/z} \;\Rightarrow\; u_x = \dfrac{y}{z}x^{(y/z)-1}$, $u_y = x^{y/z}\ln x\cdot\dfrac{1}{z} = \dfrac{x^{y/z}}{z}\ln x$, $u_z = x^{y/z}\ln x\cdot\dfrac{-y}{z^2} = -\dfrac{yx^{y/z}}{z^2}\ln x$

**31.** $f(x,y,z,t) = xyz^2\tan(yt) \;\Rightarrow\; f_x(x,y,z,t) = yz^2\tan(yt)$,
$f_y(x,y,z,t) = xyz^2\cdot\sec^2(yt)\cdot t + xz^2\tan(yt) = xyz^2t\sec^2(yt) + xz^2\tan(yt)$,
$f_z(x,y,z,t) = 2xyz\tan(yt)$, $f_t(x,y,z,t) = xyz^2\sec^2(yt)\cdot y = xy^2z^2\sec^2(yt)$.

**32.** $f(x,y,z,t) = \dfrac{xy^2}{t+2z} \;\Rightarrow$

$f_x(x,y,z,t) = \dfrac{y^2}{t+2z}$, $f_y(x,y,z,t) = \dfrac{2xy}{t+2z}$,

$f_z(x,y,z,t) = xy^2(-1)(t+2z)^{-2}(2) = -\dfrac{2xy^2}{(t+2z)^2}$, $f_t(x,y,z,t) = xy^2(-1)(t+2z)^{-2}(1) = -\dfrac{xy^2}{(t+2z)^2}$.

**33.** $u = \sqrt{x_1^2+x_2^2+\cdots+x_n^2}$. For each $i = 1,\ldots,n$,
$u_{x_i} = \frac{1}{2}\left(x_1^2+x_2^2+\cdots+x_n^2\right)^{-1/2}(2x_i) = \dfrac{x_i}{\sqrt{x_1^2+x_2^2+\cdots+x_n^2}}$.

**34.** $u = \sin(x_1 + 2x_2 + \cdots + nx_n)$. For each $i = 1, \ldots, n$, $u_{x_i} = i\cos(x_1 + 2x_2 + \cdots + nx_n)$.

**35.** $f(x,y) = \sqrt{x^2+y^2} \quad\Rightarrow\quad f_x(x,y) = \frac{1}{2}(x^2+y^2)^{-1/2}(2x) = \dfrac{x}{\sqrt{x^2+y^2}}$, so $f_x(3,4) = \dfrac{3}{\sqrt{3^2+4^2}} = \dfrac{3}{5}$.

**36.** $f(x,y) = \sin(2x+3y) \quad\Rightarrow\quad f_y(x,y) = \cos(2x+3y)\cdot 3 = 3\cos(2x+3y)$, so
$f_y(-6,4) = 3\cos[2(-6)+3(4)] = 3\cos 0 = 3$.

**37.** $f(x,y,z) = \dfrac{x}{y+z} = x(y+z)^{-1} \quad\Rightarrow\quad f_z(x,y,z) = x(-1)(y+z)^{-2} = -\dfrac{x}{(y+z)^2}$, so
$f_z(3,2,1) = -\dfrac{3}{(2+1)^2} = -\dfrac{1}{3}$.

**38.** $f(u,v,w) = w\tan(uv) \quad\Rightarrow\quad f_v(u,v,w) = w\sec^2(uv)\cdot u = uw\sec^2(uv)$, so
$f_v(2,0,3) = (2)(3)\sec^2(2\cdot 0) = 6$.

**39.** $f(x,y) = x^2 - xy + 2y^2 \quad\Rightarrow$

$$\begin{aligned} f_x(x,y) &= \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h} = \lim_{h\to 0}\frac{(x+h)^2-(x+h)y+2y^2-(x^2-xy+2y^2)}{h} \\ &= \lim_{h\to 0}\frac{h(2x-y+h)}{h} = \lim_{h\to 0}(2x-y+h) = 2x-y \end{aligned}$$

$$\begin{aligned} f_y(x,y) &= \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h} = \lim_{h\to 0}\frac{x^2-x(y+h)+2(y+h)^2-(x^2-xy+2y^2)}{h} \\ &= \lim_{h\to 0}\frac{h(4y-x+2h)}{h} = \lim_{h\to 0}(4y-x+2h) = 4y-x \end{aligned}$$

**40.** $f(x,y) = \sqrt{3x-y} \quad\Rightarrow$

$$\begin{aligned} f_x(x,y) &= \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h} \\ &= \lim_{h\to 0}\frac{\sqrt{3(x+h)-y}-\sqrt{3x-y}}{h}\cdot\frac{\sqrt{3(x+h)-y}+\sqrt{3x-y}}{\sqrt{3(x+h)-y}+\sqrt{3x-y}} \\ &= \lim_{h\to 0}\frac{3}{\sqrt{3(x+h)-y}+\sqrt{3x-y}} = \frac{3}{2\sqrt{3x-y}} \end{aligned}$$

$$\begin{aligned} f_y(x,y) &= \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h} \\ &= \lim_{h\to 0}\frac{\sqrt{3x-(y+h)}-\sqrt{3x-y}}{h}\cdot\frac{\sqrt{3x-(y+h)}+\sqrt{3x-y}}{\sqrt{3x-(y+h)}+\sqrt{3x-y}} \\ &= \lim_{h\to 0}\frac{-1}{\sqrt{3x-(y+h)}+\sqrt{3x-y}} = \frac{-1}{2\sqrt{3x-y}} \end{aligned}$$

**41.** $x^2+y^2+z^2 = 3xyz \quad\Rightarrow\quad \dfrac{\partial}{\partial x}(x^2+y^2+z^2) = \dfrac{\partial}{\partial x}(3xyz) \quad\Rightarrow\quad 2x+0+2z\dfrac{\partial z}{\partial x} = 3y\left(x\dfrac{\partial z}{\partial x}+z\cdot 1\right)$
$\Leftrightarrow\quad 2z\dfrac{\partial z}{\partial x} - 3xy\dfrac{\partial z}{\partial x} = 3yz-2x \quad\Leftrightarrow\quad (2z-3xy)\dfrac{\partial z}{\partial x} = 3yz-2x$, so $\dfrac{\partial z}{\partial x} = \dfrac{3yz-2x}{2z-3xy}$.

$\frac{\partial}{\partial y}(x^2+y^2+z^2)=\frac{\partial}{\partial y}(3xyz) \Rightarrow 0+2y+2z\frac{\partial z}{\partial y}=3x\left(y\frac{\partial z}{\partial y}+z\cdot 1\right) \Leftrightarrow$

$2z\frac{\partial z}{\partial y}-3xy\frac{\partial z}{\partial y}=3xz-2y \Leftrightarrow (2z-3xy)\frac{\partial z}{\partial y}=3xz-2y$, so $\frac{\partial z}{\partial y}=\frac{3xz-2y}{2z-3xy}$.

**42.** $yz=\ln(x+z) \Rightarrow \frac{\partial}{\partial x}(yz)=\frac{\partial}{\partial x}(\ln(x+z)) \Rightarrow y\frac{\partial z}{\partial x}=\frac{1}{x+z}\left(1+\frac{\partial z}{\partial x}\right) \Leftrightarrow$

$\left(y-\frac{1}{x+z}\right)\frac{\partial z}{\partial x}=\frac{1}{x+z}$, so $\frac{\partial z}{\partial x}=\frac{1/(x+z)}{y-1/(x+z)}=\frac{1}{y(x+z)-1}$.

$\frac{\partial}{\partial y}(yz)=\frac{\partial}{\partial y}(\ln(x+z)) \Rightarrow y\frac{\partial z}{\partial y}+z\cdot 1=\frac{1}{x+z}\left(0+\frac{\partial z}{\partial y}\right) \Leftrightarrow \left(y-\frac{1}{x+z}\right)\frac{\partial z}{\partial y}=-z$, so

$\frac{\partial z}{\partial y}=\frac{-z}{y-1/(x+z)}=\frac{z(x+z)}{1-y(x+z)}$.

**43.** $x-z=\arctan(yz) \Rightarrow \frac{\partial}{\partial x}(x-z)=\frac{\partial}{\partial x}(\arctan(yz)) \Rightarrow 1-\frac{\partial z}{\partial x}=\frac{1}{1+(yz)^2}\cdot y\frac{\partial z}{\partial x} \Leftrightarrow$

$1=\left(\frac{y}{1+y^2z^2}+1\right)\frac{\partial z}{\partial x} \Leftrightarrow 1=\left(\frac{y+1+y^2z^2}{1+y^2z^2}\right)\frac{\partial z}{\partial x}$, so $\frac{\partial z}{\partial x}=\frac{1+y^2z^2}{1+y+y^2z^2}$.

$\frac{\partial}{\partial y}(x-z)=\frac{\partial}{\partial y}(\arctan(yz)) \Rightarrow 0-\frac{\partial z}{\partial y}=\frac{1}{1+(yz)^2}\cdot\left(y\frac{\partial z}{\partial y}+z\cdot 1\right) \Leftrightarrow$

$-\frac{z}{1+y^2z^2}=\left(\frac{y}{1+y^2z^2}+1\right)\frac{\partial z}{\partial y} \Leftrightarrow -\frac{z}{1+y^2z^2}=\left(\frac{y+1+y^2z^2}{1+y^2z^2}\right)\frac{\partial z}{\partial y} \Leftrightarrow \frac{\partial z}{\partial y}=-\frac{z}{1+y+y^2z^2}$.

**44.** $\sin(xyz)=x+2y+3z \Rightarrow \frac{\partial}{\partial x}(\sin(xyz))=\frac{\partial}{\partial x}(x+2y+3z) \Rightarrow$

$\cos(xyz)\cdot y\left(x\frac{\partial z}{\partial x}+z\right)=1+3\frac{\partial z}{\partial x} \Leftrightarrow (xy\cos(xyz)-3)\frac{\partial z}{\partial x}=1-yz\cos(xyz)$, so

$\frac{\partial z}{\partial x}=\frac{1-yz\cos(xyz)}{xy\cos(xyz)-3}$.

$\frac{\partial}{\partial y}(\sin(xyz))=\frac{\partial}{\partial y}(x+2y+3z) \Rightarrow \cos(xyz)\cdot x\left(y\frac{\partial z}{\partial y}+z\right)=2+3\frac{\partial z}{\partial y} \Leftrightarrow$

$(xy\cos(xyz)-3)\frac{\partial z}{\partial y}=2-xz\cos(xyz)$, so $\frac{\partial z}{\partial y}=\frac{2-xz\cos(xyz)}{xy\cos(xyz)-3}$.

**45.** (a) $z=f(x)+g(y) \Rightarrow \frac{\partial z}{\partial x}=f'(x), \frac{\partial z}{\partial y}=g'(y)$

(b) $z=f(x+y)$. Let $u=x+y$. Then $\frac{\partial z}{\partial x}=\frac{df}{du}\frac{\partial u}{\partial x}=\frac{df}{du}(1)=f'(u)=f'(x+y)$,

$\frac{\partial z}{\partial y}=\frac{df}{du}\frac{\partial u}{\partial y}=\frac{df}{du}(1)=f'(u)=f'(x+y)$.