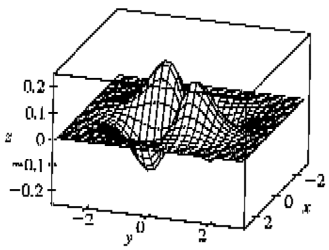
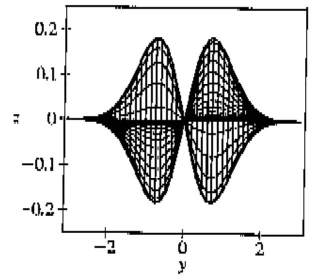
the values of $f$ there are larger than at the neighboring points. There does not appear to be any local minimum point; although the valley shape between the two peaks looks like a minimum of some kind, some neighboring points have lower function values.

**66.** $f(x,y) = xye^{-x^2-y^2}$



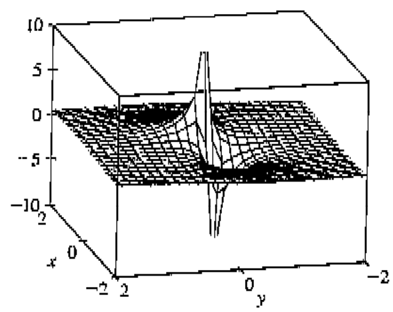

Three-dimensional view




Front view

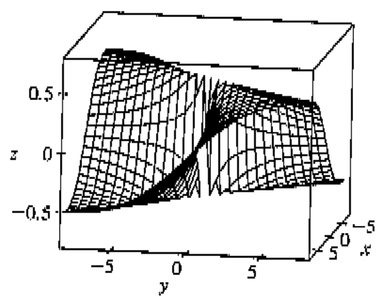
The function does have a maximum value, which it appears to achieve at two different points (the two "hilltops"). From the front view graph, we can estimate the maximum value to be approximately 0.18. These same two points can also be considered local maximum points. The two "valley bottoms" visible in the graph can be considered local minimum points, as all the neighboring points give greater values of $f$.

**67.**



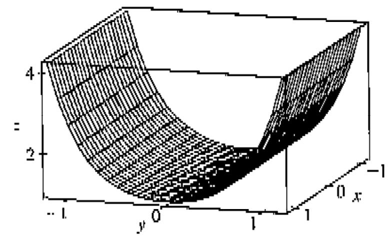

$f(x,y) = \dfrac{x+y}{x^2+y^2}$. As both $x$ and $y$ become large, the function values appear to approach 0, regardless of which direction is considered. As $(x,y)$ approaches the origin, the graph exhibits asymptotic behavior. From some directions, $f(x,y) \to \infty$, while in others $f(x,y) \to -\infty$. (These are the vertical spikes visible in the graph.) If the graph is examined carefully, however, one can see that $f(x,y)$ approaches 0 along the line $y = -x$.

**68.**



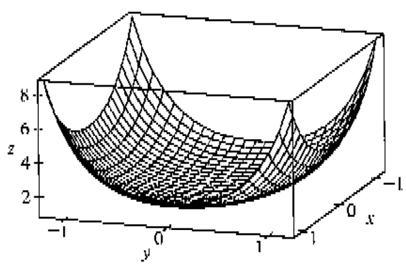

$f(x,y) = \dfrac{xy}{x^2+y^2}$. The graph exhibits different limiting values as $x$ and $y$ become large or as $(x,y)$ approaches the origin, depending on the direction being examined. For example, although $f$ is undefined at the origin, the function values appear to be $\frac{1}{2}$ along the line $y = x$, regardless of the distance from the origin. Along the line $y = -x$, the value is always $-\frac{1}{2}$. Along the axes, $f(x,y) = 0$ for all values of $(x,y)$ except the origin. Other directions, heading toward the origin or away from the origin, give various limiting values between $-\frac{1}{2}$ and $\frac{1}{2}$.

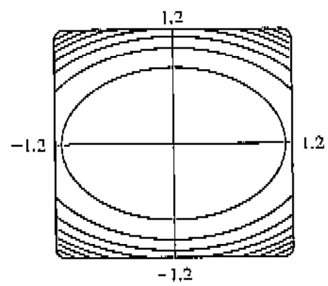
**69.** $f(x, y) = e^{cx^2+y^2}$. First, if $c = 0$, the graph is the cylindrical surface $z = e^{y^2}$ (whose level curves are parallel lines). When $c > 0$, the vertical trace above the $y$-axis remains fixed while the sides of the surface in the $x$-direction "curl" upward, giving the graph a shape resembling an elliptic paraboloid. The level curves of the surface are ellipses centered at the origin.




$c = 0$

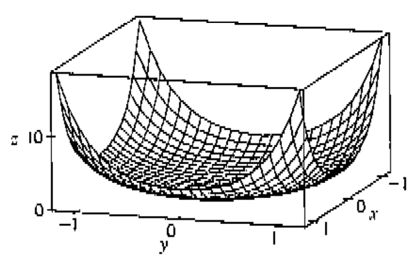
For $0 < c < 1$, the ellipses have major axis the $x$-axis and the eccentricity increases as $c \to 0$.

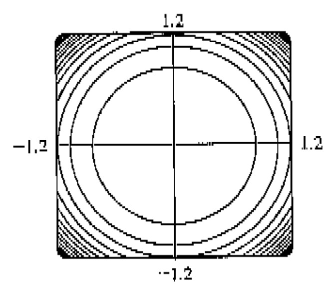



$c = 0.5$ (level curves in increments of 1)

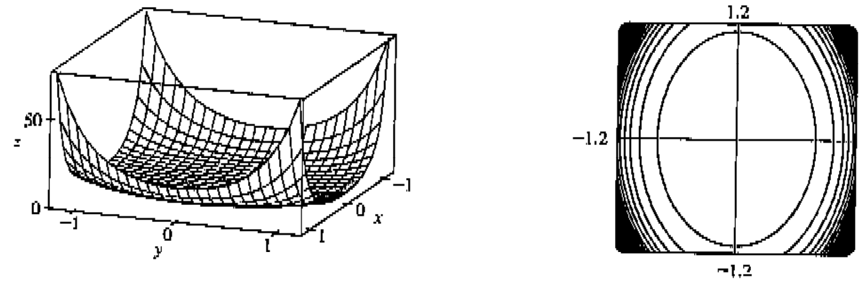
For $c = 1$ the level curves are circles centered at the origin.



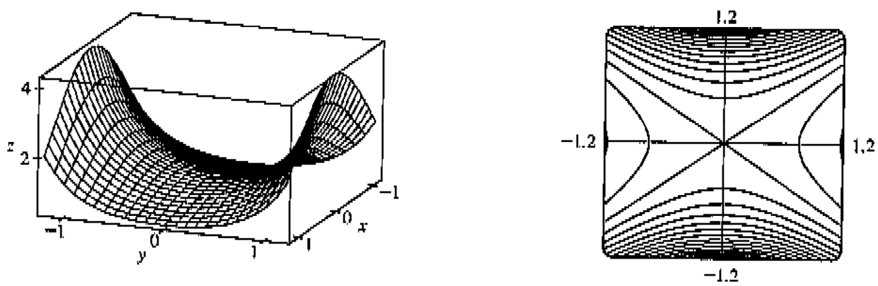

$c = 1$ (level curves in increments of 1)

When $c > 1$, the level curves are ellipses with major axis the $y$-axis, and the eccentricity increases as $c$ increases.

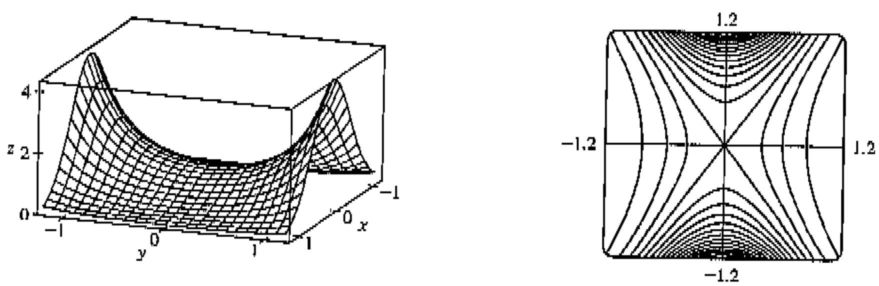



$c = 2$ (level curves in increments of 4)

For values of $c < 0$, the sides of the surface in the $x$-direction curl downward and approach the $xy$-plane (while the vertical trace $x = 0$ remains fixed), giving a saddle-shaped appearance to the graph near the point $(0, 0, 1)$. The level curves consist of a family of hyperbolas. As $c$ decreases, the surface becomes flatter in the $x$-direction and the surface's approach to the curve in the trace $x = 0$ becomes steeper, as the graphs demonstrate.

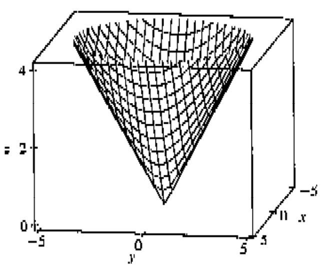



$c = -0.5$ (level curves in increments of 0.25)

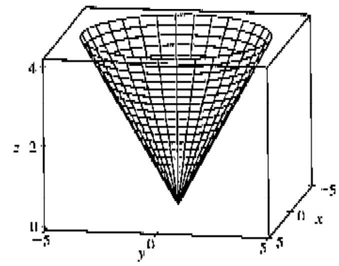



$c = -2$ (level curves in increments of 0.25)

**70.** First, we graph $f(x, y) = \sqrt{x^2 + y^2}$. As an alternative, the $x^2 + y^2$ expression suggests that cylindrical coordinates may be appropriate, giving the equivalent equation $z = \sqrt{r^2} = r$, $r \geq 0$ which we graph as well. Notice that the graph in cylindrical coordinates better demonstrates the symmetry of the surface.

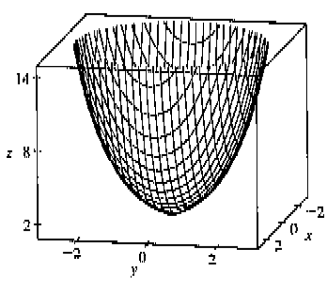


$f(x, y) = \sqrt{x^2 + y^2}$

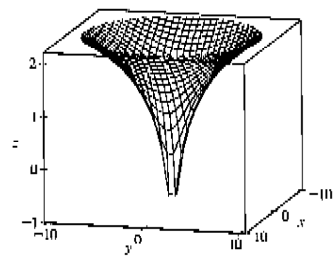


$z = r$, $r \geq 0$

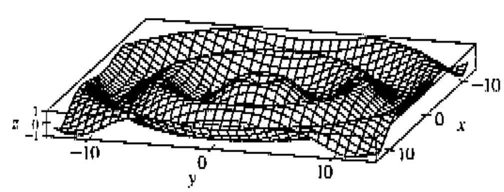
Graphs of the other four functions follow.



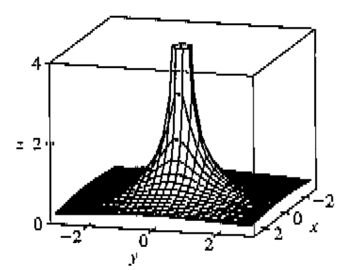
$f(x, y) = e^{\sqrt{x^2 + y^2}}$



$f(x, y) = \ln \sqrt{x^2 + y^2}$



$f(x, y) = \sin\left(\sqrt{x^2 + y^2}\right)$



$f(x, y) = \dfrac{1}{\sqrt{x^2 + y^2}}$

Notice that each graph $f(x, y) = g\left(\sqrt{x^2 + y^2}\right)$ exhibits radial symmetry about the $z$-axis and the trace in the $xz$-plane for $x \geq 0$ is the graph of $z = g(x)$, $x \geq 0$. This suggests that the graph of $f(x, y) = g\left(\sqrt{x^2 + y^2}\right)$ is obtained from the graph of $g$ by graphing $z = g(x)$ in the $xz$-plane and rotating the curve about the $z$-axis.

**71.** (a) $P = bL^{\alpha}K^{1-\alpha} \Rightarrow \dfrac{P}{K} = bL^{\alpha}K^{-\alpha} \Rightarrow \dfrac{P}{K} = b\left(\dfrac{L}{K}\right)^{\alpha} \Rightarrow \ln\dfrac{P}{K} = \ln\left(b\left(\dfrac{L}{K}\right)^{\alpha}\right) \Rightarrow$

$\ln\dfrac{P}{K} = \ln b + \alpha\ln\left(\dfrac{L}{K}\right)$

(b) We list the values for $\ln(L/K)$ and $\ln(P/K)$ for the years 1899–1922. (Historically, these values were rounded to 2 decimal places.)

| Year | $x = \ln(L/K)$ | $y = \ln(P/K)$ |
|---|---|---|
| 1899 | 0 | 0 |
| 1900 | −0.02 | −0.06 |
| 1901 | −0.04 | −0.02 |
| 1902 | −0.04 | 0 |
| 1903 | −0.07 | −0.05 |
| 1904 | −0.13 | −0.12 |
| 1905 | −0.18 | −0.04 |
| 1906 | −0.20 | −0.07 |
| 1907 | −0.23 | −0.15 |
| 1908 | −0.41 | −0.38 |
| 1909 | −0.33 | −0.24 |
| 1910 | −0.35 | −0.27 |

| Year | $x = \ln(L/K)$ | $y = \ln(P/K)$ |
|---|---|---|
| 1911 | −0.38 | −0.34 |
| 1912 | −0.38 | −0.24 |
| 1913 | −0.41 | −0.25 |
| 1914 | −0.47 | −0.37 |
| 1915 | −0.53 | −0.34 |
| 1916 | −0.49 | −0.28 |
| 1917 | −0.53 | −0.39 |
| 1918 | −0.60 | −0.50 |
| 1919 | −0.68 | −0.57 |
| 1920 | −0.74 | −0.57 |
| 1921 | −1.05 | −0.85 |
| 1922 | −0.98 | −0.59 |

After entering the $(x, y)$ pairs into a calculator or CAS, the resulting least squares regression line through the points is approximately $y = 0.75136x + 0.01053$, which we round to $y = 0.75x + 0.01$.

(c) Comparing the regression line from part (b) to the equation $y = \ln b + \alpha x$ with $x = \ln(L/K)$ and $y = \ln(P/K)$, we have $\alpha = 0.75$ and $\ln b = 0.01 \Rightarrow b = e^{0.01} \approx 1.01$. Thus, the Cobb-Douglas production function is $P = bL^{\alpha}K^{1-\alpha} = 1.01L^{0.75}K^{0.25}$.

## 15.2 Limits and Continuity ET 14.2

**1.** In general, we can't say anything about $f(3,1)$! $\lim_{(x,y)\to(3,1)} f(x,y) = 6$ means that the values of $f(x,y)$ approach 6 as $(x,y)$ approaches, but is not equal to, $(3,1)$. If $f$ is continuous, we know that $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$, so $\lim_{(x,y)\to(3,1)} f(x,y) = f(3,1) = 6$.

**2.** (a) The outdoor temperature as a function of longitude, latitude, and time is continuous. Small changes in longitude, latitude, or time can produce only small changes in temperature, as the temperature doesn't jump abruptly from one value to another.

(b) Elevation is not necessarily continuous. If we think of a cliff with a sudden drop-off, a very small change in longitude or latitude can produce a comparatively large change in elevation, without all the intermediate values being attained. Elevation *can* jump from one value to another.

(c) The cost of a taxi ride is usually discontinuous. The cost normally increases in jumps, so small changes in distance traveled or time can produce a jump in cost. A graph of the function would show breaks in the surface.

**3.** We make a table of values of $f(x,y) = \dfrac{x^2y^3 + x^3y^2 - 5}{2 - xy}$ for a set of $(x,y)$ points near the origin.

| x \ y | −0.2 | −0.1 | −0.05 | 0 | 0.05 | 0.1 | 0.2 |
|---|---|---|---|---|---|---|---|
| −0.2 | −2.551 | −2.525 | −2.513 | −2.500 | −2.488 | −2.475 | −2.451 |
| −0.1 | −2.525 | −2.513 | −2.506 | −2.500 | −2.494 | −2.488 | −2.475 |
| −0.05 | −2.513 | −2.506 | −2.503 | −2.500 | −2.497 | −2.494 | −2.488 |
| 0 | −2.500 | −2.500 | −2.500 | | −2.500 | −2.500 | −2.500 |
| 0.05 | −2.488 | −2.494 | −2.497 | −2.500 | −2.503 | −2.506 | −2.513 |
| 0.1 | −2.475 | −2.488 | −2.494 | −2.500 | −2.506 | −2.513 | −2.525 |
| 0.2 | −2.451 | −2.475 | −2.488 | −2.500 | −2.513 | −2.525 | −2.551 |

As the table shows, the values of $f(x,y)$ seem to approach $-2.5$ as $(x,y)$ approaches the origin from a variety of different directions. This suggests that $\lim_{(x,y)\to(0,0)} f(x,y) = -2.5$.

Since $f$ is a rational function, it is continuous on its domain. $f$ is defined at $(0,0)$, so we can use direct substitution to establish that $\lim_{(x,y)\to(0,0)} f(x,y) = \dfrac{0^2 0^3 + 0^3 0^2 - 5}{2 - 0 \cdot 0} = -\dfrac{5}{2}$, verifying our guess.

**4.** We make a table of values of $f(x,y) = \dfrac{2xy}{x^2 + 2y^2}$ for a set of $(x,y)$ points near the origin.

| x \ y | −0.3 | −0.2 | −0.1 | 0 | 0.1 | 0.2 | 0.3 |
|---|---|---|---|---|---|---|---|
| −0.3 | 0.667 | 0.706 | 0.545 | 0.000 | −0.545 | −0.706 | −0.667 |
| −0.2 | 0.545 | 0.667 | 0.667 | 0.000 | −0.667 | −0.667 | −0.545 |
| −0.1 | 0.316 | 0.444 | 0.667 | 0.000 | −0.667 | −0.444 | −0.316 |
| 0 | 0.000 | 0.000 | 0.000 | | 0.000 | 0.000 | 0.000 |
| 0.1 | −0.316 | −0.444 | −0.667 | 0.000 | 0.667 | 0.444 | 0.316 |
| 0.2 | −0.545 | −0.667 | −0.667 | 0.000 | 0.667 | 0.667 | 0.545 |
| 0.3 | −0.667 | −0.706 | −0.545 | 0.000 | 0.545 | 0.706 | 0.667 |

It appears from the table that the values of $f(x,y)$ are not approaching a single value as $(x,y)$ approaches the origin. For verification, if we first approach $(0,0)$ along the $x$-axis, we have $f(x,0) = 0$, so $f(x,y) \to 0$. But if we approach $(0,0)$ along the line $y = x$, $f(x,x) = \dfrac{2x^2}{x^2 + 2x^2} = \dfrac{2}{3}$ $(x \neq 0)$, so $f(x,y) \to \frac{2}{3}$. Since $f$ approaches different values along different paths to the origin, this limit does not exist.

**5.** $f(x,y) = x^5 + 4x^3y - 5xy^2$ is a polynomial, and hence continuous, so $\lim_{(x,y)\to(5,-2)} f(x,y) = f(5,-2) = 5^5 + 4(5)^3(-2) - 5(5)(-2)^2 = 2025$.

**6.** $x - 2y$ is a polynomial and therefore continuous. Since $\cos t$ is a continuous function, the composition $\cos(x - 2y)$ is also continuous. $xy$ is also a polynomial, and hence continuous, so the product $f(x,y) = xy\cos(x - 2y)$ is a continuous function. Then $\lim_{(x,y)\to(6,3)} f(x,y) = f(6,3) = (6)(3)\cos(6 - 2 \cdot 3) = 18$.