15 ☐ PARTIAL DERIVATIVES ☐ ET 14

15.1 Functions of Several Variables ET 14.1

1. (a) From Table 1, $f(-15, 40) = -27$, which means that if the temperature is $-15°C$ and the wind speed is 40 km/h, then the air would feel equivalent to approximately $-27°C$ without wind.

(b) The question is asking: when the temperature is $-20°C$, what wind speed gives a wind-chill index of $-30°C$? From Table 1, the speed is 20 km/h.

(c) The question is asking: when the wind speed is 20 km/h, what temperature gives a wind-chill index of $-49°C$? From Table 1, the temperature is $-35°C$.

(d) The function $W = f(-5, v)$ means that we fix T at -5 and allow v to vary, resulting in a function of one variable. In other words, the function gives wind-chill index values for different wind speeds when the temperature is $-5°C$. From Table 1 (look at the row corresponding to $T = -5$), the function decreases and appears to approach a constant value as v increases.

(e) The function $W = f(T, 50)$ means that we fix v at 50 and allow T to vary, again giving a function of one variable. In other words, the function gives wind-chill index values for different temperatures when the wind speed is 50 km/h . From Table 1 (look at the column corresponding to $v = 50$), the function increases almost linearly as T increases.

2. (a) From the table, $f(95, 70) = 124$, which means that when the actual temperature is $95°F$ and the relative humidity is 70%, the perceived air temperature is approximately $124°F$.

(b) Looking at the row corresponding to $T = 90$, we see that $f(90, h) = 100$ when $h = 60$.

(c) Looking at the column corresponding to $h = 50$, we see that $f(T, 50) = 88$ when $T = 85$.

(d) $I = f(80, h)$ means that T is fixed at 80 and h is allowed to vary, resulting in a function of h that gives the humidex values for different relative humidities when the actual temperature is $80°F$. Similarly, $I = f(100, h)$ is a function of one variable that gives the humidex values for different relative humidities when the actual temperature is $100°F$. Looking at the rows of the table corresponding to $T = 80$ and $T = 100$, we see that $f(80, h)$ increases at a relatively constant rate of approximately $1°F$ per 10% relative humidity, while $f(100, h)$ increases more quickly (at first with an average rate of change of $5°F$ per 10% relative humidity) and at an increasing rate (approximately $12°F$ per 10% relative humidity for larger values of h).

3. If the amounts of labor and capital are both doubled, we replace L, K in the function with $2L, 2K$, giving

$$P(2L, 2K) = 1.01(2L)^{0.75}(2K)^{0.25} = 1.01(2^{0.75})(2^{0.25})L^{0.75}K^{0.25} = (2^1)1.01L^{0.75}K^{0.25}$$
$$= 2P(L, K)$$

Thus, the production is doubled. It is also true for the general case $P(L, K) = bL^{\alpha}K^{1-\alpha}$:

$$P(2L, 2K) = b(2L)^{\alpha}(2K)^{1-\alpha} = b(2^{\alpha})(2^{1-\alpha})L^{\alpha}K^{1-\alpha} = (2^{\alpha+1-\alpha})bL^{\alpha}K^{1-\alpha} = 2P(L, K).$$

4. We compare the values for the wind-chill index given by Table 1 with those given by the model function:

Modeled Wind-Chill Index Values $W(T, v)$

Wind Speed (km/h)

Actual temperature (°C)

T \ V	5	10	15	20	25	30	40	50	60	70	80
5	4.08	2.66	1.74	1.07	0.52	0.05	−0.71	−1.33	−1.85	−2.30	−2.70
0	−1.59	−3.31	−4.42	−5.24	−5.91	−6.47	−7.40	−8.14	−8.77	−9.32	−9.80
−5	−7.26	−9.29	−10.58	−11.55	−12.34	−13.00	−14.08	−14.96	−15.70	−16.34	−16.91
−10	−12.93	−15.26	−16.75	−17.86	−18.76	−19.52	−20.77	−21.77	−22.62	−23.36	−24.01
−15	−18.61	−21.23	−22.91	−24.17	−25.19	−26.04	−27.45	−28.59	−29.54	−30.38	−31.11
−20	−24.28	−27.21	−29.08	−30.48	−31.61	−32.57	−34.13	−35.40	−36.47	−37.40	−38.22
−25	−29.95	−33.18	−35.24	−36.79	−38.04	−39.09	−40.82	−42.22	−43.39	−44.42	−45.32
−30	−35.62	−39.15	−41.41	−43.10	−44.46	−45.62	−47.50	−49.03	−50.32	−51.44	−52.43
−35	−41.30	−45.13	−47.57	−49.41	−50.89	−52.14	−54.19	−55.84	−57.24	−58.46	−59.53
−40	−46.97	−51.10	−53.74	−55.72	−57.31	−58.66	−60.87	−62.66	−64.17	−65.48	−66.64

The values given by the function appear to be fairly close (within 0.5) to the values in Table 1.

5. (a) According to the table, $f(40, 15) = 25$, which means that if a 40-knot wind has been blowing in the open sea for 15 hours, it will create waves with estimated heights of 25 feet.

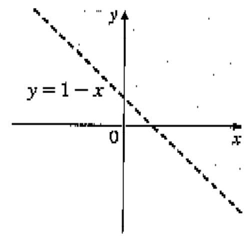
(b) $h = f(30, t)$ means we fix v at 30 and allow t to vary, resulting in a function of one variable. Thus here, $h = f(30, t)$ gives the wave heights produced by 30-knot winds blowing for t hours. From the table (look at the row corresponding to $v = 30$), the function increases but at a declining rate as t increases. In fact, the function values appear to be approaching a limiting value of approximately 19, which suggests that 30-knot winds cannot produce waves higher than about 19 feet.

(c) $h = f(v, 30)$ means we fix t at 30, again giving a function of one variable. So, $h = f(v, 30)$ gives the wave heights produced by winds of speed v blowing for 30 hours. From the table (look at the column corresponding to $t = 30$), the function appears to increase at an increasing rate, with no apparent limiting value. This suggests that faster winds (lasting 30 hours) always create higher waves.

6. (a) $f(1,1) = \ln(1+1-1) = \ln 1 = 0$

(b) $f(e,1) = \ln(e+1-1) = \ln e = 1$

(c) $\ln(x+y-1)$ is defined only when $x+y-1>0$, that is, $y > 1-x$. So the domain of f is $\{(x,y) \mid y > 1-x\}$.



(d) Since $\ln(x+y-1)$ can be any real number, the range is $\mathbb{R}$.

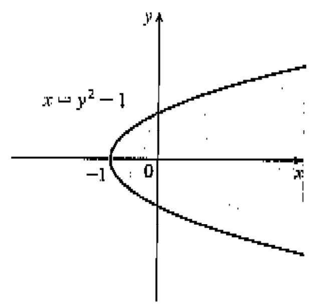
7. (a) $f(2,0) = 2^2 e^{3(2)(0)} = 4(1) = 4$

(b) Since both x^2 and the exponential function are defined everywhere, $x^2 e^{3xy}$ is defined for all choices of values for x and y. Thus, the domain of f is $\mathbb{R}^2$.

(c) Because the range of $g(x,y) = 3xy$ is $\mathbb{R}$, and the range of e^x is $(0,\infty)$, the range of $e^{g(x,y)} = e^{3xy}$ is $(0,\infty)$. The range of x^2 is $[0,\infty)$, so the range of the product $x^2 e^{3xy}$ is $[0,\infty)$.

8. $\sqrt{1+x-y^2}$ is defined only when $1+x-y^2 \geq 0 \;\Rightarrow\; x \geq y^2 - 1$, so the domain of f is $\{(x,y) \mid x \geq y^2-1\}$, all those points on or to the right of the parabola $x = y^2 - 1$.

The range of f is $[0,\infty)$.



9. (a) $f(2,-1,6) = e^{\sqrt{6-2^2-(-1)^2}} = e^{\sqrt{1}} = e$.

(b) $e^{\sqrt{z-x^2-y^2}}$ is defined when $z - x^2 - y^2 \geq 0 \;\Rightarrow\; z \geq x^2 + y^2$. Thus the domain of f is $\{(x,y,z) \mid z \geq x^2+y^2\}$.

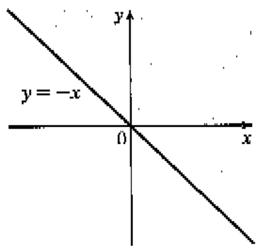
(c) Since $\sqrt{z-x^2-y^2} \geq 0$, we have $e^{\sqrt{z-x^2-y^2}} \geq 1$. Thus the range of f is $[1,\infty)$.

10. (a) $g(2,-2,4) = \ln(25 - 2^2 - (-2)^2 - 4^2) = \ln 1 = 0$.

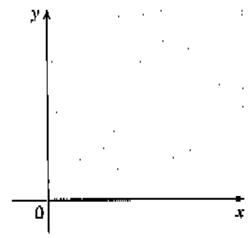
(b) For the logarithmic function to be defined, we need $25 - x^2 - y^2 - z^2 > 0$. Thus the domain of g is $\{(x,y,z) \mid x^2+y^2+z^2 < 25\}$, the interior of the sphere $x^2+y^2+z^2 = 25$.

(c) Since $0 < 25 - x^2 - y^2 - z^2 \leq 25$ for (x,y,z) in the domain of g, $\ln(25-x^2-y^2-z^2) \leq \ln 25$. Thus the range of g is $(-\infty, \ln 25]$.

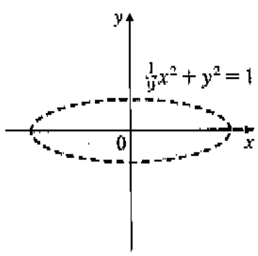
11. $\sqrt{x+y}$ is defined only when $x + y \geq 0$, or $y \geq -x$. So the domain of f is $\{(x, y) \mid y \geq -x\}$.



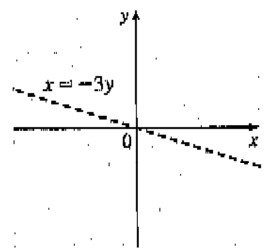
12. We need $x \geq 0$ and $y \geq 0$, so $D = \{(x, y) \mid x \geq 0 \text{ and } y \geq 0\}$, the first quadrant.



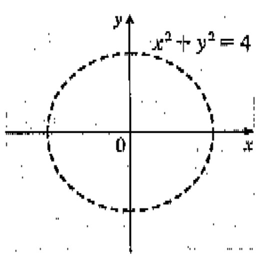
13. $\ln\left(9 - x^2 - 9y^2\right)$ is defined only when $9 - x^2 - 9y^2 > 0$, or $\frac{1}{9}x^2 + y^2 < 1$. So the domain of f is $\left\{(x, y) \mid \frac{1}{9}x^2 + y^2 < 1\right\}$, the interior of an ellipse.



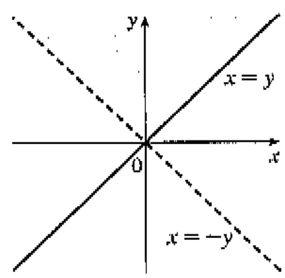
14. $\dfrac{x - 3y}{x + 3y}$ is defined only when $x + 3y \neq 0$, or $x \neq -3y$. So the domain of f is $\{(x, y) \mid x \neq -3y\}$.



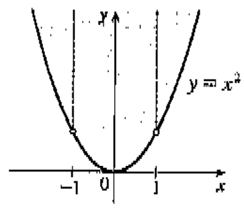
15. $\dfrac{3x + 5y}{x^2 + y^2 - 4}$ is defined only when $x^2 + y^2 - 4 \neq 0$, or $x^2 + y^2 \neq 4$. So the domain of f is $\left\{(x, y) \mid x^2 + y^2 \neq 4\right\}$.



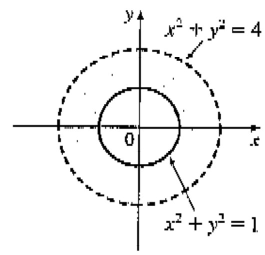
16. We need $y - x \geq 0$ or $y \geq x$ and $y + x > 0$ or $x > -y$. Thus $D = \{(x, y) \mid -y < x \leq y, y > 0\}$.



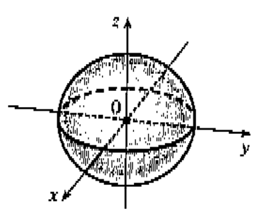
17. $\sqrt{y - x^2}$ is defined only when $y - x^2 \geq 0$, or $y \geq x^2$. In addition, f is not defined if $1 - x^2 = 0$ $\Rightarrow$ $x = \pm 1$. Thus the domain of f is $\{(x, y) \mid y \geq x^2, x \neq \pm 1\}$.



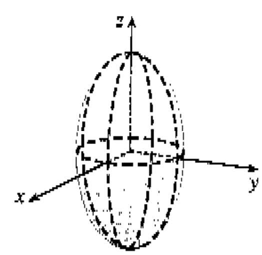
18. f is defined only when $x^2 + y^2 - 1 \geq 0$ $\Rightarrow$ $x^2 + y^2 \geq 1$ and $4 - x^2 - y^2 > 0$ $\Rightarrow$ $x^2 + y^2 < 4$. Thus $D = \{(x, y) \mid 1 \leq x^2 + y^2 < 4\}$.



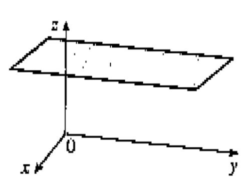
19. We need $1 - x^2 - y^2 - z^2 \geq 0$ or $x^2 + y^2 + z^2 \leq 1$, so $D = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$ (the points inside or on the sphere of radius 1, center the origin).



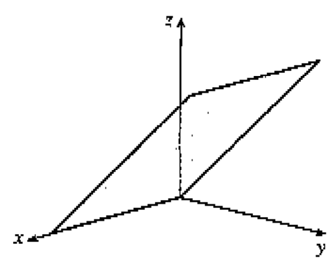
20. f is defined only when $16 - 4x^2 - 4y^2 - z^2 > 0$ $\Rightarrow$ $\frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{16} < 1$. Thus, $D = \left\{(x, y, z) \,\middle|\, \frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{16} < 1\right\}$, that is, the points inside the ellipsoid $\frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{16} = 1$.



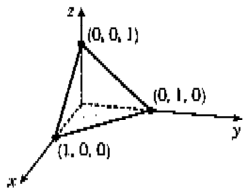
21. $z = 3$, a horizontal plane through the point $(0, 0, 3)$.



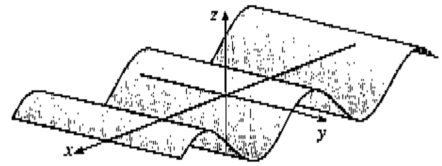
22. $z = y$, a plane which intersects the yz-plane in the line $z = y$, $x = 0$. The portion of this plane that lies in the first octant is shown.



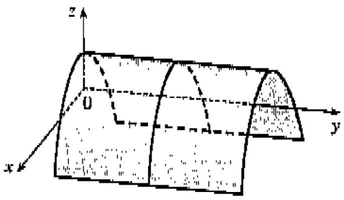
23. $z = 1 - x - y$ or $x + y + z = 1$, a plane with intercepts 1, 1, and 1.



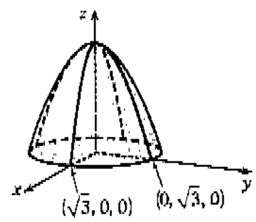
24. $z = \cos x$, a "wave."



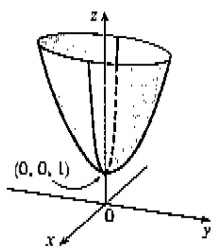
25. $z = 1 - x^2$, a parabolic cylinder.



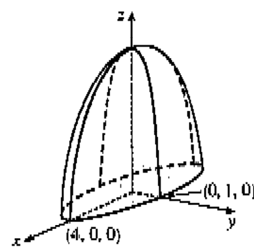
26. $z = 3 - x^2 - y^2$, a circular paraboloid with vertex at $(0, 0, 3)$.



27. $z = 4x^2 + y^2 + 1$, an elliptic paraboloid with vertex at $(0, 0, 1)$.



28. $z = \sqrt{16 - x^2 - 16y^2}$ so $z \geq 0$ and $z^2 + x^2 + 16y^2 = 16$, the top half of an ellipsoid.



29. $z = \sqrt{x^2 + y^2}$ so $x^2 + y^2 = z^2$ and $z \geq 0$, the top half of a right circular cone.