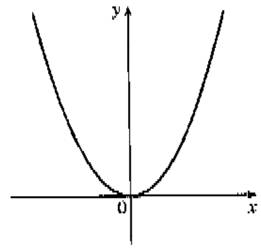
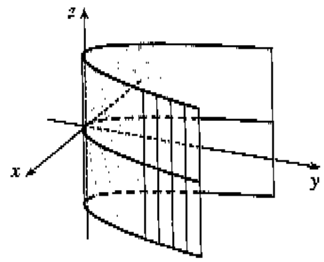
13.6 Cylinders and Quadric Surfaces

ET 12.6

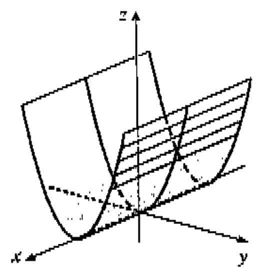
1. (a) In $\mathbb{R}^2$, the equation $y = x^2$ represents a parabola.



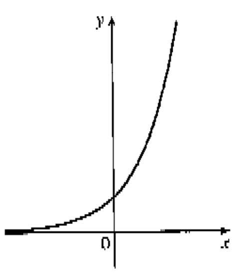
(b) In $\mathbb{R}^3$, the equation $y = x^2$ doesn't involve z, so any horizontal plane with equation $z = k$ intersects the graph in a curve with equation $y = x^2$. Thus, the surface is a parabolic cylinder, made up of infinitely many shifted copies of the same parabola. The rulings are parallel to the z-axis.



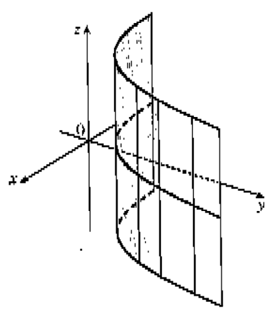
(c) In $\mathbb{R}^3$, the equation $z = y^2$ also represents a parabolic cylinder. Since x doesn't appear, the graph is formed by moving the parabola $z = y^2$ in the direction of the x-axis. Thus, the rulings of the cylinder are parallel to the x-axis.



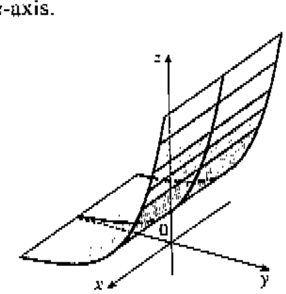
2. (a)



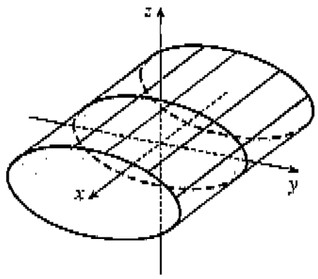
(b) Since the equation $y = e^x$ doesn't involve z, horizontal traces are copies of the curve $y = e^x$. The rulings are parallel to the z-axis.



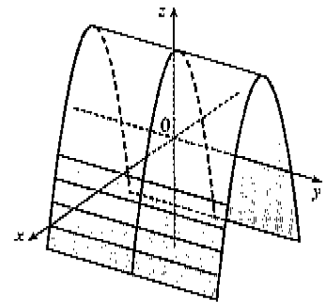
(c) The equation $z = e^y$ doesn't involve x, so vertical traces in $x = k$ (parallel to the yz-plane) are copies of the curve $z = e^y$. The rulings are parallel to the x-axis.



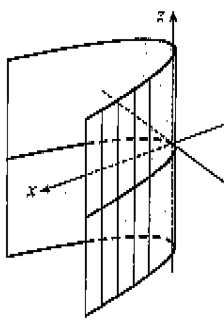
3. Since x is missing from the equation, the vertical traces $y^2 + 4z^2 = 4$, $x = k$, are copies of the same ellipse in the plane $x = k$. Thus, the surface $y^2 + 4z^2 = 4$ is an elliptic cylinder with rulings parallel to the x-axis.



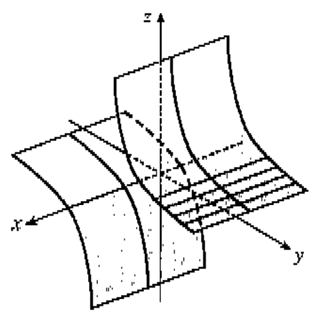
4. Since y is missing from the equation, each vertical trace $z = 4 - x^2$, $y = k$, is a copy of the same parabola in the plane $y = k$. Thus, the surface $z = 4 - x^2$ is a parabolic cylinder with rulings parallel to the y-axis.



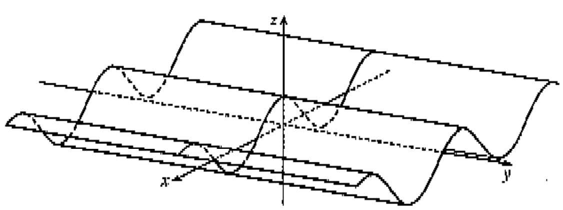
5. Since z is missing, each horizontal trace $x = y^2$, $z = k$, is a copy of the same parabola in the plane $z = k$. Thus, the surface $x - y^2 = 0$ is a parabolic cylinder with rulings parallel to the z-axis.



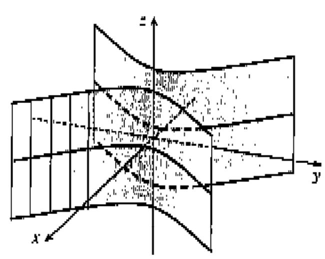
6. Since x is missing, each vertical trace $yz = 4$, $x = k$ is a copy of the same hyperbola in the plane $x = k$. Thus, the surface $yz = 4$ is a hyperbolic cylinder with rulings parallel to the x-axis.



7. Since y is missing, each vertical trace $z = \cos x$, $y = k$ is a copy of a cosine curve in the plane $y = k$. Thus, the surface $z = \cos x$ is a cylindrical surface with rulings parallel to the y-axis.

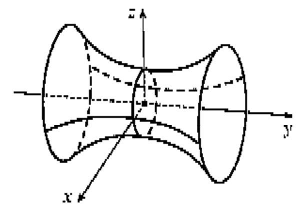


8. Since z is missing, each horizontal trace $x^2 - y^2 = 1$, $z = k$ is a copy of the same hyperbola in the plane $z = k$. Thus, the surface $x^2 - y^2 = 1$ is a hyperbolic cylinder with rulings parallel to the z-axis.

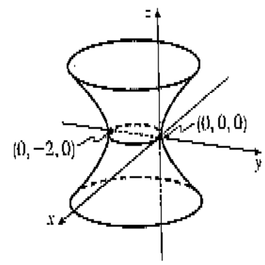


9. (a) The traces of $x^2 + y^2 - z^2 = 1$ in $x = k$ are $y^2 - z^2 = 1 - k^2$, a family of hyperbolas. (Note that the hyperbolas are oriented differently for $-1 < k < 1$ than for $k < -1$ or $k > 1$.) The traces in $y = k$ are $x^2 - z^2 = 1 - k^2$, a similar family of hyperbolas. The traces in $z = k$ are $x^2 + y^2 = 1 + k^2$, a family of circles. For $k = 0$, the trace in the xy-plane, the circle is of radius 1. As $|k|$ increases, so does the radius of the circle. This behavior, combined with the hyperbolic vertical traces, gives the graph of the hyperboloid of one sheet in Table 1.

(b) The shape of the surface is unchanged, but the hyperboloid is rotated so that its axis is the y-axis. Traces in $y = k$ are circles, while traces in $x = k$ and $z = k$ are hyperbolas.

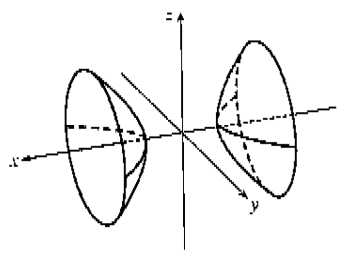


(c) Completing the square in y gives $x^2 + (y + 1)^2 - z^2 = 1$. The surface is a hyperboloid identical to the one in part (a) but shifted one unit in the negative y-direction.

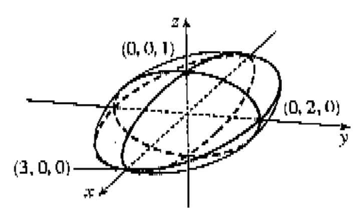


10. (a) The traces of $-x^2 - y^2 + z^2 = 1$ in $x = k$ are $-y^2 + z^2 = 1 + k^2$, a family of hyperbolas, as are the traces in $y = k$, $-x^2 + z^2 = 1 + k^2$. The traces in $z = k$ are $x^2 + y^2 = k^2 - 1$, a family of circles for $|k| > 1$. As $|k|$ increases, the radii of the circles increase; the traces are empty for $|k| < 1$. This behavior, combined with the vertical traces, gives the graph of the hyperboloid of two sheets in Table 1.

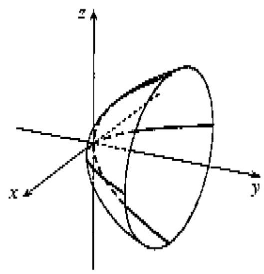
(b) The graph has the same shape as the hyperboloid in part (a) but is rotated so that its axis is the x-axis. Traces in $x = k$, $|k| > 1$, are circles, while traces in $y = k$ and $z = k$ are hyperbolas.



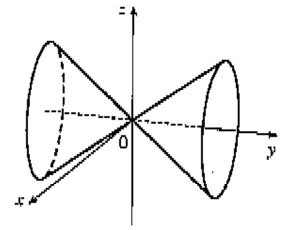
11. Traces: $x = k$, $9y^2 + 36z^2 = 36 - 4k^2$, an ellipse for $|k| < 3$; $y = k$, $4x^2 + 36z^2 = 36 - 9k^2$, an ellipse for $|k| < 2$; $z = k$, $4x^2 + 9y^2 = 36(1 - k^2)$, an ellipse for $|k| < 1$. Thus the surface is an ellipsoid with center at the origin and axes along the x-, y- and z-axes.



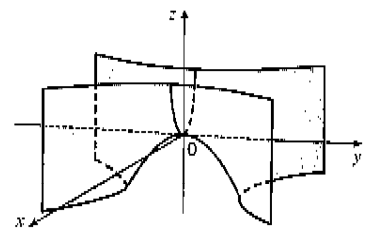
12. Traces: $x = k$, $4y = k^2 + z^2$, a parabola; $y = k$, $4k = x^2 + z^2$, a circle for $k > 0$; $z = k$, $4y = x^2 + k^2$ a parabola. Thus the surface is a circular paraboloid with axis the y-axis and vertex at $(0, 0, 0)$.



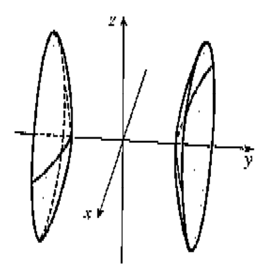
13. Traces: $x = k$, $y^2 = k^2 + z^2$ or $y^2 - z^2 = k^2$, a hyperbola for $k \neq 0$ and two intersecting lines for $k = 0$; $y = k$, $x^2 + z^2 = k^2$, a circle for $k \neq 0$; $z = k$, $y^2 = x^2 + k^2$ or $y^2 - x^2 = k^2$, a hyperbola for $k \neq 0$ and two intersecting lines for $k = 0$. Thus the surface is a cone (right circular) with axis the y-axis and vertex the origin.



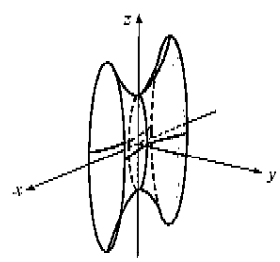
14. Traces: $x = k$, $z - k^2 = -y^2$, a parabola; $y = k$, $z + k^2 = x^2$, a parabola; $z = k$, $x^2 - y^2 = k$, a hyperbola. Thus the surface is a hyperbolic paraboloid with saddle point $(0, 0, 0)$ (and since $c > 0$, the saddle is upside down).



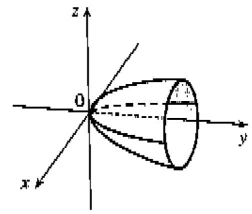
15. Traces: $x = k$, $4y^2 - z^2 = 4 + k^2$, a hyperbola; $y = k$, $x^2 + z^2 = 4k^2 - 4$, a circle for $|k| > 1$; $z = k$, $4y^2 - x^2 = 4 + k^2$, a hyperbola. Thus the surface is a hyperboloid of two sheets with axis the y-axis.



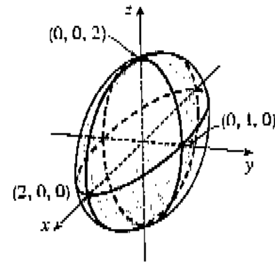
16. Traces: $x = k$, $25y^2 + z^2 = 100 + 4k^2$, an ellipse; $y = k$, $25k^2 + z^2 = 100 + 4x^2$ or $z^2 - 4x^2 = 100 - 25k^2$, a hyperbola for $|k| < 2$; $z = k$, $25y^2 + k^2 = 100 + 4x^2$ or $25y^2 - 4x^2 = 100 - k^2$, a hyperbola for $|k| < 10$. Thus the surface is a hyperboloid of one sheet with axis the x-axis.



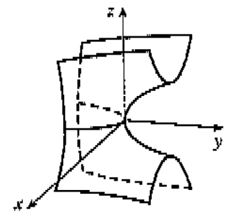
17. Traces: $x = k$, $k^2 + 4z^2 - y = 0$ or $y - k^2 = 4z^2$, a parabola; $y = k$, $x^2 + 4z^2 = k$, an ellipse for $k > 0$; $z = k$, $x^2 + 4k^2 - y = 0$ or $y - 4k^2 = x^2$, a parabola. Thus the surface is an elliptic paraboloid with axis the y-axis and vertex the origin.



18. Traces: $x = k$, $|k| \le 2 \Rightarrow y^2 + \frac{z^2}{4} = 1 - \frac{k^2}{4}$, ellipses; $y = k$, $|k| \le 1 \Rightarrow x^2 + z^2 = 4\left(1 - k^2\right)$, circles; $z = k$, $|k| \le 2 \Rightarrow \frac{x^2}{4} + y^2 = 1 - \frac{k^2}{4}$, ellipses. $x^2 + 4y^2 + z^2 = 4 \Leftrightarrow \frac{x^2}{2^2} + \frac{y^2}{1^2} + \frac{z^2}{2^2} = 1$, which is the equation of an ellipsoid.

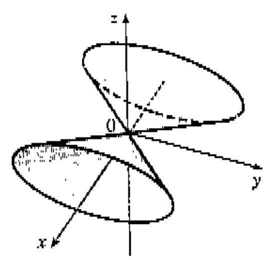


19. $y = z^2 - x^2$. The traces in $x = k$ are the parabolas $y = z^2 - k^2$; the traces in $y = k$ are $k = z^2 - x^2$, which are hyperbolas (note the hyperbolas are oriented differently for $k > 0$ than for $k < 0$); and the traces in $z = k$ are the parabolas $y = k^2 - x^2$. Thus, $\frac{y}{1} = \frac{z^2}{1^2} - \frac{x^2}{1^2}$ is a hyperbolic paraboloid.



20. Traces: $x = k \Rightarrow y^2 + 4z^2 = 16k^2$, ellipses; $y = k \Rightarrow 16x^2 - 4z^2 = k^2$, hyperbolas if $k \ne 0$ and two intersecting lines if $k = 0$; $z = k \Rightarrow 16x^2 - y^2 = 4k^2$, hyperbolas if $k \ne 0$ and two intersecting lines if $k = 0$.

$16x^2 = y^2 + 4z^2 \Leftrightarrow x^2 = \frac{y^2}{4^2} + \frac{z^2}{2^2}$ is an elliptic cone with axis the x-axis and vertex the origin.



21. This is the equation of an ellipsoid: $x^2 + 4y^2 + 9z^2 = x^2 + \frac{y^2}{(1/2)^2} + \frac{z^2}{(1/3)^2} = 1$, with x-intercepts ± 1, y-intercepts $\pm\frac{1}{2}$ and z-intercepts $\pm\frac{1}{3}$. So the major axis is the x-axis and the only possible graph is VII.

22. This is the equation of an ellipsoid: $9x^2 + 4y^2 + z^2 = \frac{x^2}{(1/3)^2} + \frac{y^2}{(1/2)^2} + z^2 = 1$, with x-intercepts $\pm\frac{1}{3}$, y-intercepts $\pm\frac{1}{2}$ and z-intercepts ± 1. So the major axis is the z-axis and the only possible graph is IV.

23. This is the equation of a hyperboloid of one sheet, with $a = b = c = 1$. Since the coefficient of y^2 is negative, the axis of the hyperboloid is the y-axis, hence the correct graph is II.

24. This is a hyperboloid of two sheets, with $a = b = c = 1$. This surface does not intersect the xz-plane at all, so the axis of the hyperboloid is the y-axis and the graph is III.

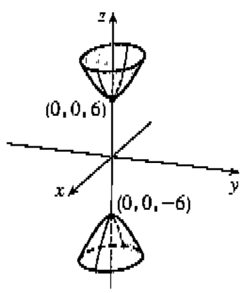
25. There are no real values of x and z that satisfy this equation for $y < 0$, so this surface does not extend to the left of the xz-plane. The surface intersects the plane $y = k > 0$ in an ellipse. Notice that y occurs to the first power whereas x and z occur to the second power. So the surface is an elliptic paraboloid with axis the y-axis. Its graph is VI.

26. This is the equation of a cone with axis the y-axis, so the graph is I.

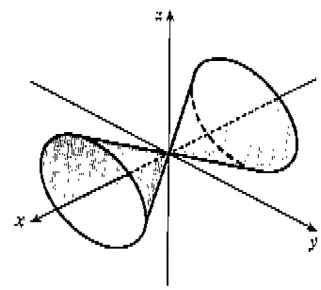
27. This surface is a cylinder because the variable y is missing from the equation. The intersection of the surface and the xz-plane is an ellipse. So the graph is VIII.

28. This is the equation of a hyperbolic paraboloid. The trace in the xy-plane is the parabola $y = x^2$. So the correct graph is V.

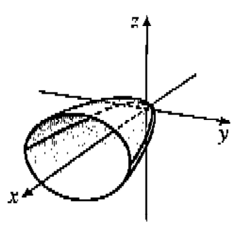
29. $z^2 = 4x^2 + 9y^2 + 36$ or $-4x^2 - 9y^2 + z^2 = 36$ or $-\frac{x^2}{9} - \frac{y^2}{4} + \frac{z^2}{36} = 1$ represents a hyperboloid of two sheets with axis the z-axis.



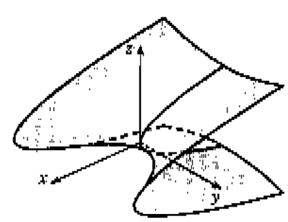
30. $x^2 = 2y^2 + 3z^2$ or $x^2 = \frac{y^2}{1/2} + \frac{z^2}{1/3}$ or $\frac{x^2}{6} = \frac{y^2}{3} + \frac{z^2}{2}$ represents an elliptic cone with vertex $(0, 0, 0)$ and axis the x-axis.



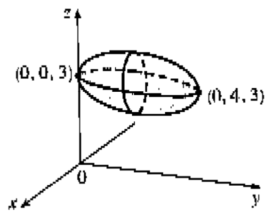
31. $x = 2y^2 + 3z^2$ or $x = \frac{y^2}{1/2} + \frac{z^2}{1/3}$ or $\frac{x}{6} = \frac{y^2}{3} + \frac{z^2}{2}$ represents an elliptic paraboloid with vertex $(0, 0, 0)$ and axis the x-axis.



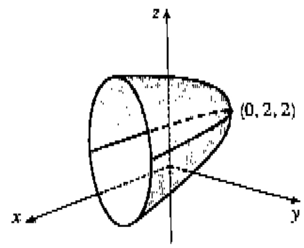
32. $4x - y^2 + 4z^2 = 0$ or $4x = y^2 - 4z^2$ or $x = \frac{y^2}{4} - z^2$ represents a hyperbolic paraboloid with center $(0, 0, 0)$.



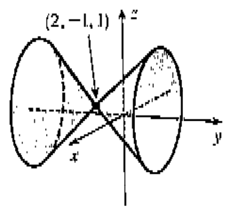
33. Completing squares in y and z gives $4x^2 + (y-2)^2 + 4(z-3)^2 = 4$ or $x^2 + \frac{(y-2)^2}{4} + (z-3)^2 = 1$, an ellipsoid with center $(0, 2, 3)$.



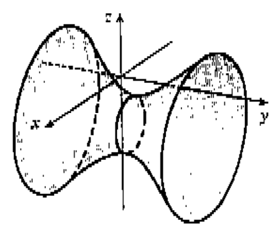
34. Completing squares in y and z gives $4(y-2)^2 + (z-2)^2 - x = 0$ or $\frac{x}{4} = (y-2)^2 + \frac{(z-2)^2}{4}$, an elliptic paraboloid with vertex $(0, 2, 2)$ and axis the horizontal line $y = 2$, $z = 2$.



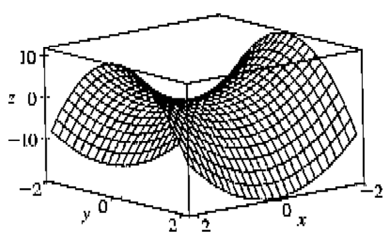
35. Completing squares in all three variables gives $(x-2)^2 - (y+1)^2 + (z-1)^2 = 0$ or $(y+1)^2 = (x-2)^2 + (z-1)^2$, a circular cone with center $(2, -1, 1)$ and axis the horizontal line $x = 2$, $z = 1$.



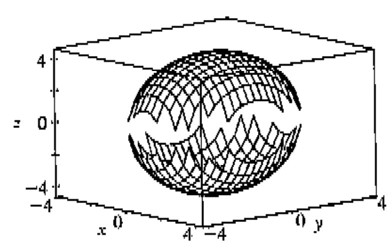
36. Completing squares in all three variables gives $(x-1)^2 - (y-1)^2 + (z+2)^2 = 2$ or $\frac{(x-1)^2}{2} - \frac{(y-1)^2}{2} + \frac{(z+2)^2}{2} = 1$, a hyperboloid of one sheet with center $(1, 1, -2)$ and axis the horizontal line $x = 1$, $z = -2$.



37.

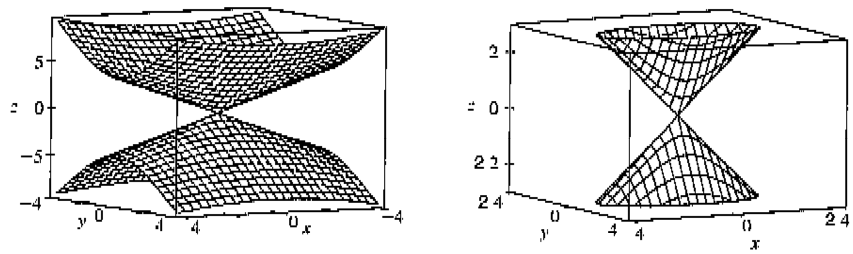


38.



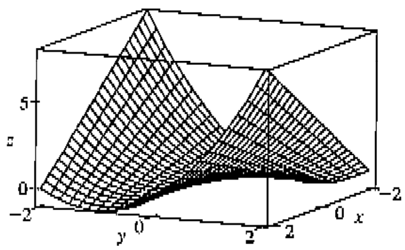
In Section 17.6 [ET 16.6], we will be able to graph ellipsoids without gaps; see Exercise 17.6.53 [ET 16.6.53].

39.

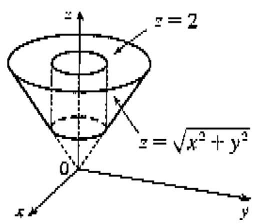


To restrict the z-range as in the second graph, we can use the option `view = -2..2` in Maple's `plot3d` command, or `PlotRange -> {-2,2}` in Mathematica's `Plot3D` command.

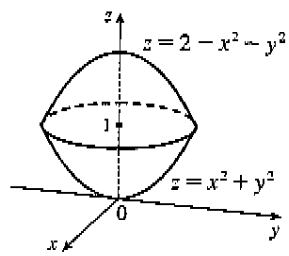
40.



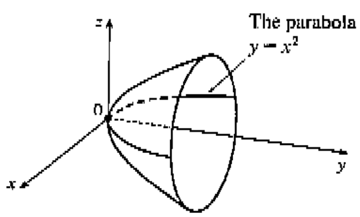
41.



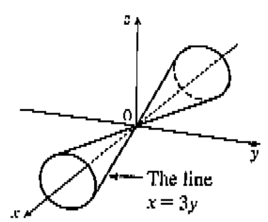
42.



43. The surface is a paraboloid of revolution (circular paraboloid) with vertex at the origin, axis the y-axis and opens to the right. Thus the trace in the yz-plane is also a parabola: $y = z^2$, $x = 0$. The equation is $y = x^2 + z^2$.



44. The surface is a right circular cone with vertex at $(0, 0, 0)$ and axis the x-axis. For $x = k \neq 0$, the trace is a circle with center $(k, 0, 0)$ and radius $r = y = \frac{x}{3} = \frac{k}{3}$. Thus the equation is $\frac{1}{9}x^2 = y^2 + z^2$.



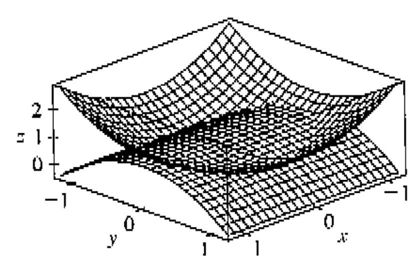
45. Let $P = (x, y, z)$ be an arbitrary point equidistant from $(-1, 0, 0)$ and the plane $x = 1$. Then the distance from P to $(-1, 0, 0)$ is $\sqrt{(x+1)^2 + y^2 + z^2}$ and the distance from P to the plane $x = 1$ is $|x - 1| / \sqrt{1^2} = |x - 1|$ (by Equation 13.5.9 [ET 12.5.9]). So $|x - 1| = \sqrt{(x+1)^2 + y^2 + z^2} \Leftrightarrow (x-1)^2 = (x+1)^2 + y^2 + z^2 \Leftrightarrow$ $x^2 - 2x + 1 = x^2 + 2x + 1 + y^2 + z^2 \Leftrightarrow -4x = y^2 + z^2$. Thus the collection of all such points P is a circular paraboloid with vertex at the origin, axis the x-axis, which opens in the negative direction.

46. Let $P = (x, y, z)$ be an arbitrary point whose distance from the x-axis is twice its distance from the yz-plane. The distance from P to the x-axis is $\sqrt{(x-x)^2 + y^2 + z^2} = \sqrt{y^2 + z^2}$ and the distance from P to the yz-plane ($x = 0$) is $|x| / 1 = |x|$. Thus $\sqrt{y^2 + z^2} = 2|x| \Leftrightarrow y^2 + z^2 = 4x^2 \Leftrightarrow x^2 = (y^2/2^2) + (z^2/2^2)$. So the surface is a right circular cone with vertex the origin and axis the x-axis.

47. If (a, b, c) satisfies $z = y^2 - x^2$, then $c = b^2 - a^2$. L_1: $x = a + t$, $y = b + t$, $z = c + 2(b - a)t$, L_2: $x = a + t$, $y = b - t$, $z = c - 2(b + a)t$. Substitute the parametric equations of L_1 into the equation of the hyperbolic paraboloid in order to find the points of intersection: $z = y^2 - x^2 \Rightarrow$ $c + 2(b - a)t = (b + t)^2 - (a + t)^2 = b^2 - a^2 + 2(b - a)t \Rightarrow c = b^2 - a^2$. As this is true for all values of t, L_1 lies on $z = y^2 - x^2$. Performing similar operations with L_2 gives: $z = y^2 - x^2 \Rightarrow$ $c - 2(b + a)t = (b - t)^2 - (a + t)^2 = b^2 - a^2 - 2(b + a)t \Rightarrow c = b^2 - a^2$. This tells us that all of L_2 also lies on $z = y^2 - x^2$.

48. Any point on the curve of intersection must satisfy both $2x^2 + 4y^2 - 2z^2 + 6x = 2$ and $2x^2 + 4y^2 - 2z^2 - 5y = 0$. Subtracting, we get $6x + 5y = 2$, which is linear and therefore the equation of a plane. Thus the curve of intersection lies in this plane.

49.



The curve of intersection looks like a bent ellipse. The projection of this curve onto the xy-plane is the set of points $(x, y, 0)$ which satisfy $x^2 + y^2 = 1 - y^2 \Leftrightarrow$ $x^2 + 2y^2 = 1 \Leftrightarrow x^2 + \dfrac{y^2}{(1/\sqrt{2})^2} = 1$. This is an equation of an ellipse.