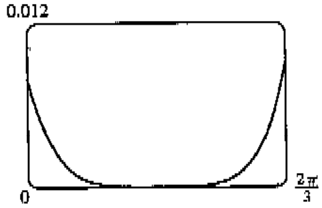
(c) 

From the graph of $|R_4(x)| = |\cos x - T_4(x)|$, it seems that the error is less than 0.01 on $\left[0, \frac{2\pi}{3}\right]$.

17.

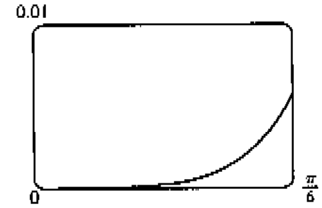
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\tan x$	0
1	$\sec^2 x$	1
2	$2\sec^2 x \tan x$	0
3	$4\sec^2 x \tan^2 x + 2\sec^4 x$	2
4	$8\sec^2 x \tan^3 x + 16\sec^4 x \tan x$	

(a) $f(x) = \tan x \approx T_3(x) = x + \frac{1}{3}x^3$

(b) $|R_3(x)| \leq \dfrac{M}{4!}|x|^4$, where $\left|f^{(4)}(x)\right| \leq M$. Now $0 \leq x \leq \frac{\pi}{6} \Rightarrow x^4 \leq \left(\frac{\pi}{6}\right)^4$, and letting $x = \frac{\pi}{6}$ $\left[\text{since } f^{(4)} \text{ is increasing on } \left(0, \frac{\pi}{6}\right)\right]$ gives

$$|R_3(x)| \leq \frac{8\left(\frac{2}{\sqrt{3}}\right)^2\left(\frac{1}{\sqrt{3}}\right)^3 + 16\left(\frac{2}{\sqrt{3}}\right)^4\left(\frac{1}{\sqrt{3}}\right)}{4!}\left(\frac{\pi}{6}\right)^4$$

$$= \frac{4\sqrt{3}}{9}\left(\frac{\pi}{6}\right)^4 \approx 0.057859$$

(c) 

From the graph of $|R_3(x)| = |\tan x - T_3(3)|$, it seems that the error is less than 0.006 on $[0, \pi/6]$.

18.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\ln(1+2x)$	$\ln 3$
1	$2/(1+2x)$	$\frac{2}{3}$
2	$-4/(1+2x)^2$	$-\frac{4}{9}$
3	$16/(1+2x)^3$	$\frac{16}{27}$
4	$-96/(1+2x)^4$	

(a) $f(x) = \ln(1+2x) \approx T_3(x) = \ln 3 + \frac{2}{3}(x-1) - \frac{4/9}{2!}(x-1)^2 + \frac{16/27}{3!}(x-1)^3$

(b) $|R_3(x)| \leq \dfrac{M}{4!}|x-1|^4$, where $\left|f^{(4)}(x)\right| \leq M$. Now $0.5 \leq x \leq 1.5 \Rightarrow -0.5 \leq x - 1 \leq 0.5 \Rightarrow |x-1| \leq 0.5 \Rightarrow |x-1|^4 \leq \frac{1}{16}$, and letting $x = 0.5$ gives $M = 6$, so $|R_3(x)| \leq \dfrac{6}{4!} \cdot \dfrac{1}{16} = \dfrac{1}{64} = 0.015625$.

15.

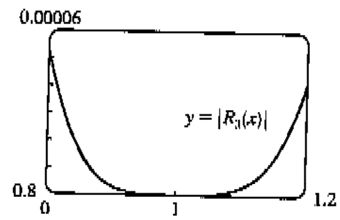
n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x^{2/3}$	1
1	$\frac{2}{3}x^{-1/3}$	$\frac{2}{3}$
2	$-\frac{2}{9}x^{-4/3}$	$-\frac{2}{9}$
3	$\frac{8}{27}x^{-7/3}$	$\frac{8}{27}$
4	$-\frac{56}{81}x^{-10/3}$	

(a) $f(x) = x^{2/3} \approx T_3(x) = 1 + \frac{2}{3}(x-1) - \frac{2/9}{2!}(x-1)^2 + \frac{8/27}{3!}(x-1)^3$

$$= 1 + \tfrac{2}{3}(x-1) - \tfrac{1}{9}(x-1)^2 + \tfrac{4}{81}(x-1)^3$$

(b) $|R_3(x)| \le \frac{M}{4!}|x-1|^4$, where $\left|f^{(4)}(x)\right| \le M$. Now $0.8 \le x \le 1.2 \;\Rightarrow\; |x-1| \le 0.2 \;\Rightarrow$ $|x-1|^4 \le 0.0016$. Since $\left|f^{(4)}(x)\right|$ is decreasing on $[0.8, 1.2]$, we can take $M = \left|f^{(4)}(0.8)\right| = \frac{56}{81}(0.8)^{-10/3}$, so $|R_3(x)| \le \dfrac{\frac{56}{81}(0.8)^{-10/3}}{24}(0.0016) \approx 0.000\,096\,97$.

(c) From the graph of $|R_3(x)| = \left|x^{2/3} - T_3(x)\right|$, it seems that the error is less than $0.000\,053\,3$ on $[0.8, 1.2]$.



16.

n	$f^{(n)}(x)$	$f^{(n)}(\frac{\pi}{3})$
0	$\cos x$	$\frac{1}{2}$
1	$-\sin x$	$-\frac{\sqrt{3}}{2}$
2	$-\cos x$	$-\frac{1}{2}$
3	$\sin x$	$\frac{\sqrt{3}}{2}$
4	$\cos x$	$\frac{1}{2}$
5	$-\sin x$	

(a) $f(x) = \cos x \approx T_4(x)$

$$= \tfrac{1}{2} - \tfrac{\sqrt{3}}{2}\left(x - \tfrac{\pi}{3}\right) - \tfrac{1}{4}\left(x - \tfrac{\pi}{3}\right)^2 + \tfrac{\sqrt{3}}{12}\left(x - \tfrac{\pi}{3}\right)^3 + \tfrac{1}{48}\left(x - \tfrac{\pi}{3}\right)^4$$

(b) $|R_4(x)| \le \dfrac{M}{5!}\left|x - \frac{\pi}{3}\right|^5$, where $\left|f^{(5)}(x)\right| \le M$. Now $0 \le x \le \frac{2\pi}{3} \;\Rightarrow\; \left(x - \frac{\pi}{3}\right)^5 \le \left(\frac{\pi}{3}\right)^5$, and letting $x = \frac{\pi}{2}$ gives $M = 1$, so $|R_4(x)| \le \frac{1}{5!}\left(\frac{\pi}{3}\right)^5 \approx 0.0105$.

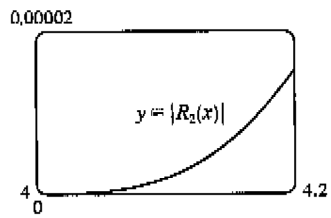
13.

n	$f^{(n)}(x)$	$f^{(n)}(4)$
0	$\sqrt{x}$	2
1	$\frac{1}{2}x^{-1/2}$	$\frac{1}{4}$
2	$-\frac{1}{4}x^{-3/2}$	$-\frac{1}{32}$
3	$\frac{3}{8}x^{-5/2}$	

(a) $f(x) = \sqrt{x} \approx T_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1/32}{2!}(x-4)^2 = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$

(b) $|R_2(x)| \le \dfrac{M}{3!}|x-4|^3$, where $|f'''(x)| \le M$. Now $4 \le x \le 4.2 \;\Rightarrow\; |x-4| \le 0.2 \;\Rightarrow$ $|x-4|^3 \le 0.008$. Since $f'''(x)$ is decreasing on $[4, 4.2]$, we can take $M = |f'''(4)| = \frac{3}{8}4^{-5/2} = \frac{3}{256}$, so $|R_2(x)| \le \frac{3/256}{6}(0.008) = \frac{0.008}{512} = 0.000015625$.

(c) From the graph of $|R_2(x)| = |\sqrt{x} - T_2(x)|$, it seems that the error is less than 1.52×10^{-5} on $[4, 4.2]$.



14.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	x^{-2}	1
1	$-2x^{-3}$	-2
2	$6x^{-4}$	6
3	$-24x^{-5}$	

(a) $f(x) = x^{-2} \approx T_2(x)$

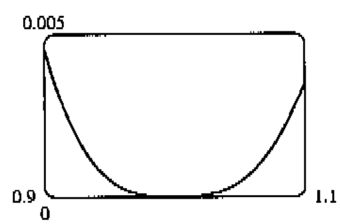
$= 1 - 2(x-1) + \frac{6}{2!}(x-1)^2$

$= 1 - 2(x-1) + 3(x-1)^2$

(b) $|R_2(x)| \le \dfrac{M}{3!}|x-1|^3$, where $|f'''(x)| \le M$. Now $0.9 \le x \le 1.1 \;\Rightarrow\; |x-1| \le 0.1 \;\Rightarrow$ $|x-1|^3 \le 0.001$. Since $f'''(x)$ is decreasing on $[0.9, 1.1]$, we can take $M = |f'''(0.9)| = \frac{24}{(0.9)^5}$, so $|R_2(x)| \le \frac{24/(0.9)^5}{6}(0.001) = \frac{0.004}{0.59049}$

≈ 0.00677404

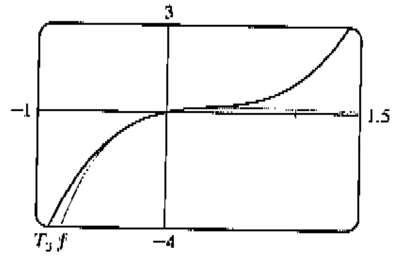
(c)



From the graph of $|R_2(x)| = |x^{-2} - T_2(x)|$, it seems that the error is less than 0.0046 on $[0.9, 1.1]$.

9.

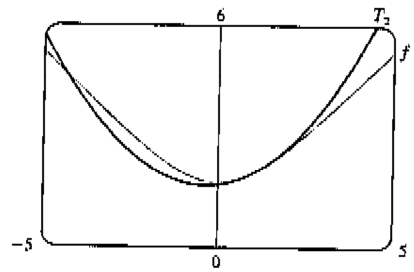
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	xe^{-2x}	0
1	$(1-2x)e^{-2x}$	1
2	$4(x-1)e^{-2x}$	-4
3	$4(3-2x)e^{-2x}$	12



$$T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(0)}{n!}x^n = \tfrac{0}{1}\cdot 1 + \tfrac{1}{1}x^1 + \tfrac{-4}{2}x^2 + \tfrac{12}{6}x^3 = x - 2x^2 + 2x^3$$

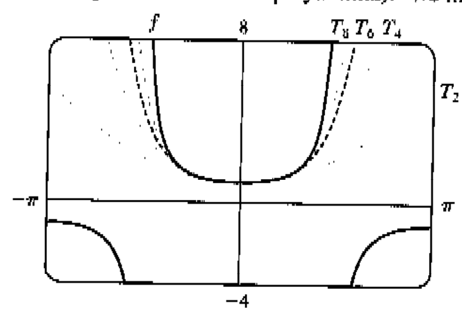
10.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$(3+x^2)^{1/2}$	2
1	$x(3+x^2)^{-1/2}$	$\frac{1}{2}$
2	$3(3+x^2)^{-3/2}$	$\frac{3}{8}$

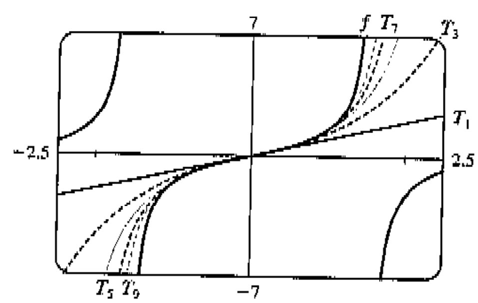


$$T_2(x) = \sum_{n=0}^{2} \frac{f^{(n)}(1)}{n!}(x-1)^n = 2 + \tfrac{1}{2}(x-1) + \tfrac{3/8}{2}(x-1)^2 = 2 + \tfrac{1}{2}(x-1) + \tfrac{3}{16}(x-1)^2$$

11. In Maple, we can find the Taylor polynomials by the following method: first define `f:=sec(x);` and then set `T2:=convert(taylor(f,x=0,3),polynom);`, `T4:=convert(taylor(f,x=0,5),polynom);`, etc. (The third argument in the `taylor` function is one more than the degree of the desired polynomial). We must `convert` to the type `polynom` because the output of the `taylor` function contains an error term which we do not want. In Mathematica, we use `Tn:=Normal[Series[f,{x,0,n}]]`, with `n=2, 4`, etc. Note that in Mathematica, the "degree" argument is the same as the degree of the desired polynomial. In Derive, author $\sec x$, then enter `Calculus,Taylor,8,0`; and then simplify the expression. The eighth Taylor polynomial is $T_8(x) = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{277}{8064}x^8$.

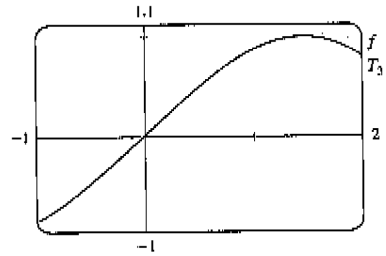


12. See Exercise 11 for the CAS commands used to generate the Taylor polynomials. The ninth Taylor polynomial for $\tan x$ is $T_9(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9$.



5.

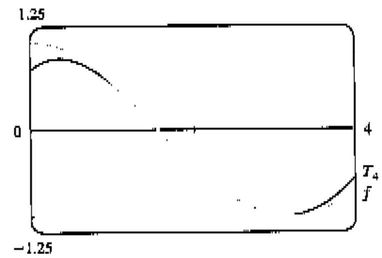
n	$f^{(n)}(x)$	$f^{(n)}(\frac{\pi}{6})$
0	$\sin x$	$\frac{1}{2}$
1	$\cos x$	$\frac{\sqrt{3}}{2}$
2	$-\sin x$	$-\frac{1}{2}$
3	$-\cos x$	$-\frac{\sqrt{3}}{2}$



$$T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(\frac{\pi}{6})}{n!}(x - \tfrac{\pi}{6})^n = \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}(x - \tfrac{\pi}{6}) - \tfrac{1}{4}(x - \tfrac{\pi}{6})^2 - \tfrac{\sqrt{3}}{12}(x - \tfrac{\pi}{6})^3$$

6.

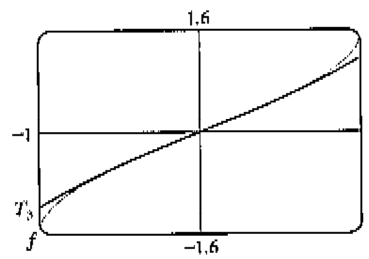
n	$f^{(n)}(x)$	$f^{(n)}(\frac{2\pi}{3})$
0	$\cos x$	$-\frac{1}{2}$
1	$-\sin x$	$-\frac{\sqrt{3}}{2}$
2	$-\cos x$	$\frac{1}{2}$
3	$\sin x$	$\frac{\sqrt{3}}{2}$
4	$\cos x$	$-\frac{1}{2}$



$$T_4(x) = \sum_{n=0}^{4} \frac{f^{(n)}(\frac{2\pi}{3})}{n!}(x - \tfrac{2\pi}{3})^n = -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}(x - \tfrac{2\pi}{3}) + \tfrac{1}{4}(x - \tfrac{2\pi}{3})^2 + \tfrac{\sqrt{3}}{12}(x - \tfrac{2\pi}{3})^3 - \tfrac{1}{48}(x - \tfrac{2\pi}{3})^4$$

7.

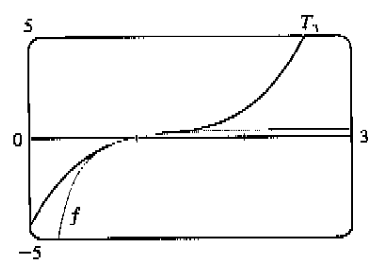
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\arcsin x$	0
1	$1/\sqrt{1-x^2}$	1
2	$x/(1-x^2)^{3/2}$	0
3	$(2x^2+1)/(1-x^2)^{5/2}$	1



$$T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(0)}{n!}x^n = x + \tfrac{x^3}{6}$$

8.

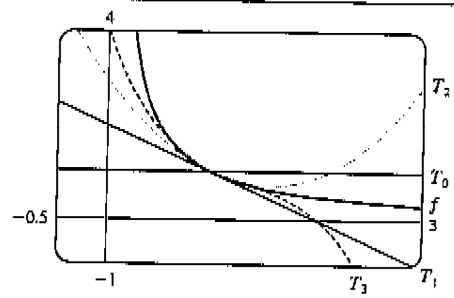
n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$(\ln x)/x$	0
1	$(1-\ln x)/x^2$	1
2	$(-3+2\ln x)/x^3$	-3
3	$(11-6\ln x)/x^4$	11



$$T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(1)}{n!}(x-1)^n = (x-1) - \tfrac{3}{2}(x-1)^2 + \tfrac{11}{6}(x-1)^3$$

2. (a)

n	$f^{(n)}(x)$	$f^{(n)}(1)$	$T_n(x)$
0	x^{-1}	1	1
1	$-x^{-2}$	-1	$1-(x-1)=2-x$
2	$2x^{-3}$	2	$1-(x-1)+(x-1)^2=x^2-3x+3$
3	$-6x^{-4}$	-6	$1-(x-1)+(x-1)^2-(x-1)^3=-x^3+4x^2-6x+4$



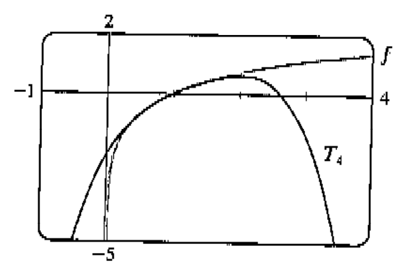
(b)

x	f	T_0	T_1	T_2	T_3
0.9	$1.\overline{1}$	1	1.1	1.11	1.111
1.3	0.7692	1	0.7	0.79	0.763

(c) As n increases, $T_n(x)$ is a good approximation to $f(x)$ on a larger and larger interval.

3.

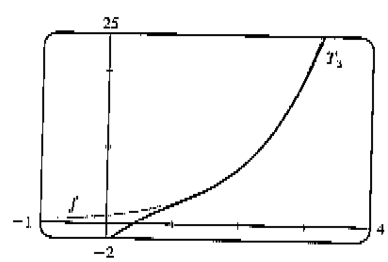
n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\ln x$	0
1	$1/x$	1
2	$-1/x^2$	-1
3	$2/x^3$	2
4	$-6/x^4$	-6



$$T_4(x)=\sum_{n=0}^{4}\frac{f^{(n)}(1)}{n!}(x-1)^n=0+(x-1)-\tfrac{1}{2}(x-1)^2+\tfrac{1}{3}(x-1)^3-\tfrac{1}{4}(x-1)^4$$

4.

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	e^x	e^2
1	e^x	e^2
2	e^x	e^2
3	e^x	e^2



$$T_3(x)=\sum_{n=0}^{3}\frac{f^{(n)}(2)}{n!}(x-2)^n=e^2+e^2(x-2)+\frac{e^2}{2}(x-2)^2+\frac{e^2}{6}(x-2)^3$$

$$L = 4a\int_0^{\pi/2}\sqrt{1-e^2\sin^2\theta}\,d\theta = 4a\int_0^{\pi/2}\left(1-\frac{1}{2}e^2\sin^2\theta - \sum_{n=2}^{\infty}\frac{1\cdot3\cdot5\cdot\cdots\cdot(2n-3)}{2^n\cdot n!}e^{2n}\sin^{2n}\theta\right)d\theta$$

$$= 4a\left[\frac{\pi}{2} - \frac{e^2}{2}S_1 - \sum_{n=2}^{\infty}\frac{1\cdot3\cdot5\cdot\cdots\cdot(2n-3)}{n!}\left(\frac{e^2}{2}\right)^n S_n\right]$$

where $S_n = \int_0^{\pi/2}\sin^{2n}\theta\,d\theta = \frac{1\cdot3\cdot5\cdot\cdots\cdot(2n-1)}{2\cdot4\cdot6\cdot\cdots\cdot2n}\frac{\pi}{2}$ by Exercise 44 of 8.1 [ET 7.1].

$$L = 4a\left(\frac{\pi}{2}\right)\left[1-\frac{e^2}{2}\cdot\frac{1}{2}-\sum_{n=2}^{\infty}\frac{1\cdot3\cdot5\cdot\cdots\cdot(2n-3)}{n!}\left(\frac{e^2}{2}\right)^n\frac{1\cdot3\cdot5\cdot\cdots\cdot(2n-1)}{2\cdot4\cdot6\cdot\cdots\cdot2n}\right]$$

$$= 2\pi a\left[1-\frac{e^2}{4}-\sum_{n=2}^{\infty}\frac{e^{2n}}{2^n}\cdot\frac{1^2\cdot3^2\cdot5^2\cdot\cdots\cdot(2n-3)^2(2n-1)}{n!\cdot2^n\cdot n!}\right]$$

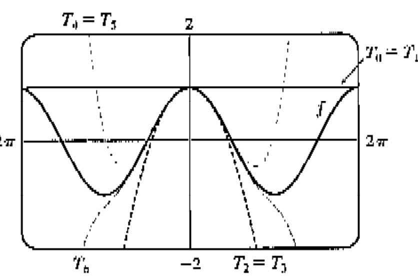
$$= 2\pi a\left[1-\frac{e^2}{4}-\sum_{n=2}^{\infty}\frac{e^{2n}}{4^n}\left(\frac{1\cdot3\cdot\cdots\cdot(2n-3)}{n!}\right)^2(2n-1)\right]$$

$$= 2\pi a\left[1-\frac{e^2}{4}-\frac{3e^4}{64}-\frac{5e^6}{256}-\cdots\right] = \frac{\pi a}{128}(256-64e^2-12e^4-5e^6-\cdots)$$

12.12 Applications of Taylor Polynomials ET 11.12

1. (a)

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$T_n(x)$
0	$\cos x$	1	1
1	$-\sin x$	0	1
2	$-\cos x$	-1	$1-\frac{1}{2}x^2$
3	$\sin x$	0	$1-\frac{1}{2}x^2$
4	$\cos x$	1	$1-\frac{1}{2}x^2+\frac{1}{24}x^4$
5	$-\sin x$	0	$1-\frac{1}{2}x^2+\frac{1}{24}x^4$
6	$-\cos x$	-1	$1-\frac{1}{2}x^2+\frac{1}{24}x^4-\frac{1}{720}x^6$



(b)

x	f	$T_0 = T_1$	$T_2 = T_3$	$T_4 = T_5$	T_6
$\frac{\pi}{4}$	0.7071	1	0.6916	0.7074	0.7071
$\frac{\pi}{2}$	0	1	-0.2337	0.0200	-0.0009
π	-1	1	-3.9348	0.1239	-1.2114

(c) As n increases, $T_n(x)$ is a good approximation to $f(x)$ on a larger and larger interval.

19. (a) $g(x) = \sum_{n=0}^{\infty} \binom{k}{n} x^n \Rightarrow g'(x) = \sum_{n=1}^{\infty} \binom{k}{n} n x^{n-1}$, so

$$(1+x)g'(x) = (1+x)\sum_{n=1}^{\infty}\binom{k}{n} n x^{n-1} = \sum_{n=1}^{\infty}\binom{k}{n} n x^{n-1} + \sum_{n=1}^{\infty}\binom{k}{n} n x^{n}$$

$$= \sum_{n=0}^{\infty}\binom{k}{n+1}(n+1)x^n + \sum_{n=0}^{\infty}\binom{k}{n} n x^n \qquad \left[\begin{array}{c}\text{Replace } n \text{ with } n+1 \\ \text{in the first series}\end{array}\right]$$

$$= \sum_{n=0}^{\infty}(n+1)\frac{k(k-1)(k-2)\cdots(k-n+1)(k-n)}{(n+1)!}x^n$$

$$+ \sum_{n=0}^{\infty}\left[(n)\frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}\right]x^n$$

$$= \sum_{n=0}^{\infty}\frac{(n+1)k(k-1)(k-2)\cdots(k-n+1)}{(n+1)!}\left[(k-n)+n\right]x^n$$

$$= k\sum_{n=0}^{\infty}\frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}x^n = k\sum_{n=0}^{\infty}\binom{k}{n}x^n = kg(x)$$

Thus, $g'(x) = \dfrac{kg(x)}{1+x}$.

(b) $h(x) = (1+x)^{-k}g(x) \Rightarrow$

$$h'(x) = -k(1+x)^{-k-1}g(x) + (1+x)^{-k}g'(x) \quad \text{[Product Rule]}$$

$$= -k(1+x)^{-k-1}g(x) + (1+x)^{-k}\frac{kg(x)}{1+x} \quad \text{[from part (a)]}$$

$$= -k(1+x)^{-k-1}g(x) + k(1+x)^{-k-1}g(x) = 0$$

(c) From part (b) we see that $h(x)$ must be constant for $x \in (-1,1)$, so $h(x) = h(0) = 1$ for $x \in (-1,1)$.

Thus, $h(x) = 1 = (1+x)^{-k}g(x) \Leftrightarrow g(x) = (1+x)^k$ for $x \in (-1,1)$.

20. By Exercise 12.11.1 [ET 11.11.1], $\sqrt{1+x} = 1 + \dfrac{x}{2} + \sum_{n=2}^{\infty}\dfrac{(-1)^{n-1}1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)x^n}{2^n\cdot n!}$, so

$(1-x^2)^{1/2} = 1 - \dfrac{1}{2}x^2 - \sum_{n=2}^{\infty}\dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^n\cdot n!}x^{2n}$ and

$\sqrt{1-e^2\sin^2\theta} = 1 - \dfrac{1}{2}e^2\sin^2\theta - \sum_{n=2}^{\infty}\dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^n\cdot n!}e^{2n}\sin^{2n}\theta$. Thus,