39. $0.123\overline{456} = \dfrac{123}{1000} + \dfrac{0.000456}{1-0.001} = \dfrac{123}{1000} + \dfrac{456}{999{,}000} = \dfrac{123{,}333}{999{,}000} = \dfrac{41{,}111}{333{,}000}$

40. $5.\overline{6021} = 5 + \dfrac{6021}{10^4} + \dfrac{6021}{10^8} + \cdots = 5 + \dfrac{6021/10^4}{1-1/10^4} = 5 + \dfrac{6021}{9999} = \dfrac{56{,}016}{9999} = \dfrac{6224}{1111}$

41. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{3^n} = \sum_{n=1}^{\infty}\left(\frac{x}{3}\right)^n$ is a geometric series with $r = \dfrac{x}{3}$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $\dfrac{|x|}{3} < 1$ $\Leftrightarrow$ $|x| < 3$; that is, $-3 < x < 3$. In that case, the sum of the series is $\dfrac{a}{1-r} = \dfrac{x/3}{1-x/3} = \dfrac{x/3}{1-x/3}\cdot\dfrac{3}{3} = \dfrac{x}{3-x}$.

42. $\sum_{n=1}^{\infty}(x-4)^n$ is a geometric series with $r = x-4$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $|x-4| < 1$ $\Leftrightarrow$ $3 < x < 5$. In that case, the sum of the series is $\dfrac{x-4}{1-(x-4)} = \dfrac{x-4}{5-x}$.

43. $\sum_{n=0}^{\infty} 4^n x^n = \sum_{n=0}^{\infty}(4x)^n$ is a geometric series with $r = 4x$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $4\,|x| < 1$ $\Leftrightarrow$ $|x| < \frac{1}{4}$. In that case, the sum of the series is $\dfrac{1}{1-4x}$.

44. $\displaystyle\sum_{n=0}^{\infty}\frac{(x+3)^n}{2^n}$ is a geometric series with $r = \dfrac{x+3}{2}$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $\dfrac{|x+3|}{2} < 1$ $\Leftrightarrow$ $|x+3| < 2$ $\Leftrightarrow$ $-5 < x < -1$. For these values of x, the sum of the series is $\dfrac{1}{1-(x+3)/2} = \dfrac{2}{2-(x+3)} = -\dfrac{2}{x+1}$.

45. $\displaystyle\sum_{n=0}^{\infty}\frac{\cos^n x}{2^n}$ is a geometric series with first term 1 and ratio $r = \dfrac{\cos x}{2}$, so it converges $\Leftrightarrow$ $|r| < 1$. But $|r| = \dfrac{|\cos x|}{2} \le \dfrac{1}{2}$ for all x. Thus, the series converges for all real values of x and the sum of the series is $\dfrac{1}{1-(\cos x)/2} = \dfrac{2}{2-\cos x}$.

46. Because $\dfrac{1}{n} \to 0$ and ln is continuous, we have $\displaystyle\lim_{n\to\infty}\ln\left(1+\frac{1}{n}\right) = \ln 1 = 0$. We now show that the series $\displaystyle\sum_{n=1}^{\infty}\ln\left(1+\frac{1}{n}\right) = \sum_{n=1}^{\infty}\ln\left(\frac{n+1}{n}\right) = \sum_{n=1}^{\infty}[\ln(n+1) - \ln n]$ diverges.
$s_n = (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \cdots + (\ln(n+1) - \ln n) = \ln(n+1) - \ln 1 = \ln(n+1)$. As $n \to \infty$, $s_n = \ln(n+1) \to \infty$, so the series diverges.

47. After defining f, We use `convert(f,parfrac);` in Maple, `Apart` in Mathematica, or `Expand Rational` and `Simplify` in Derive to find that the general term is $\dfrac{1}{(4n+1)(4n-3)} = -\dfrac{1/4}{4n+1} + \dfrac{1/4}{4n-3}$. So the

nth partial sum is

$$s_n = \sum_{k=1}^{n}\left(-\frac{1/4}{4k+1}+\frac{1/4}{4k-3}\right) = \frac{1}{4}\sum_{k=1}^{n}\left(\frac{1}{4k-3}-\frac{1}{4k+1}\right)$$
$$= \frac{1}{4}\left[\left(1-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{9}\right)+\left(\frac{1}{9}-\frac{1}{13}\right)+\cdots+\left(\frac{1}{4n-3}-\frac{1}{4n+1}\right)\right] = \frac{1}{4}\left(1-\frac{1}{4n+1}\right)$$

The series converges to $\lim_{n\to\infty} s_n = \frac{1}{4}$. This can be confirmed by directly computing the sum using `sum(f,1..infinity);` (in Maple), `Sum[f,{n,1,Infinity}]` (in Mathematica), or `Calculus Sum` (from 1 to ∞) and `Simplify` (in Derive).

48. See Exercise 47 for specific CAS commands. $\dfrac{n^2+3n+1}{(n^2+n)^2} = \dfrac{1}{n^2}+\dfrac{1}{n}-\dfrac{1}{(n+1)^2}-\dfrac{1}{n+1}$. So the nth partial sum is

$$s_n = \sum_{k=1}^{n}\left(\frac{1}{k^2}+\frac{1}{k}-\frac{1}{(k+1)^2}-\frac{1}{k+1}\right)$$
$$= \left(1+1-\frac{1}{2^2}-\frac{1}{2}\right)+\left(\frac{1}{2^2}+\frac{1}{2}-\frac{1}{3^2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n^2}+\frac{1}{n}-\frac{1}{(n+1)^2}-\frac{1}{n+1}\right)$$
$$= 1+1-\frac{1}{(n+1)^2}-\frac{1}{n+1}$$

The series converges to $\lim_{n\to\infty} s_n = 2$.

49. For $n=1$, $a_1 = 0$ since $s_1 = 0$. For $n>1$,

$$a_n = s_n - s_{n-1} = \frac{n-1}{n+1} - \frac{(n-1)-1}{(n-1)+1} = \frac{(n-1)n-(n+1)(n-2)}{(n+1)n} = \frac{2}{n(n+1)}$$

Also, $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\dfrac{1-1/n}{1+1/n} = 1$.

50. $a_1 = s_1 = 3-\frac{1}{2} = \frac{5}{2}$. For $n\neq 1$,

$$a_n = s_n - s_{n-1} = \left(3-n2^{-n}\right)-\left[3-(n-1)2^{-(n-1)}\right] = -\frac{n}{2^n}+\frac{n-1}{2^{n-1}}\cdot\frac{2}{2} = \frac{2(n-1)}{2^n}-\frac{n}{2^n} = \frac{n-2}{2^n}$$

Also, $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(3-\dfrac{n}{2^n}\right) = 3$ because $\lim_{x\to\infty}\dfrac{x}{2^x} \overset{H}{=} \lim_{x\to\infty}\dfrac{1}{2^x\ln 2} = 0$.

51. (a) The first step in the chain occurs when the local government spends D dollars. The people who receive it spend a fraction c of those D dollars, that is, Dc dollars. Those who receive the Dc dollars spend a fraction c of it, that is, Dc^2 dollars. Continuing in this way, we see that the total spending after n transactions is

$S_n = D + Dc + Dc^2 + \cdots + Dc^{n-1} = \dfrac{D(1-c^n)}{1-c}$ by (3).

(b) $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\dfrac{D(1-c^n)}{1-c} = \dfrac{D}{1-c}\lim_{n\to\infty}(1-c^n) = \dfrac{D}{1-c}$ (since $0<c<1 \Rightarrow \lim_{n\to\infty} c^n = 0$)

$= \dfrac{D}{s}$ (since $c+s=1$) $= kD$ (since $k=1/s$)

If $c=0.8$, then $s = 1-c = 0.2$ and the multiplier is $k = 1/s = 5$.

52. (a) Initially, the ball falls a distance H, then rebounds a distance rH, falls rH, rebounds r^2H, falls r^2H, etc. The total distance it travels is

$$H + 2rH + 2r^2H + 2r^3H + \cdots = H(1 + 2r + 2r^2 + 2r^3 + \cdots)$$
$$= H\left[1 + 2r(1 + r + r^2 + \cdots)\right] = H\left[1 + 2r\left(\frac{1}{1-r}\right)\right] = H\left(\frac{1+r}{1-r}\right) \text{ meters}$$

(b) From Example 3 in Section 2.1, we know that a ball falls $\frac{1}{2}gt^2$ meters in t seconds, where g is the gravitational acceleration. Thus, a ball falls h meters in $t = \sqrt{2h/g}$ seconds. The total travel time in seconds is

$$\sqrt{\frac{2H}{g}} + 2\sqrt{\frac{2H}{g}r} + 2\sqrt{\frac{2H}{g}r^2} + 2\sqrt{\frac{2H}{g}r^3} + \cdots = \sqrt{\frac{2H}{g}}\left[1 + 2\sqrt{r} + 2\sqrt{r}^2 + 2\sqrt{r}^3 + \cdots\right]$$
$$= \sqrt{\frac{2H}{g}}\left(1 + 2\sqrt{r}\left[1 + \sqrt{r} + \sqrt{r}^2 + \cdots\right]\right) = \sqrt{\frac{2H}{g}}\left[1 + 2\sqrt{r}\left(\frac{1}{1-\sqrt{r}}\right)\right] = \sqrt{\frac{2H}{g}}\frac{1+\sqrt{r}}{1-\sqrt{r}}$$

(c) It will help to make a chart of the time for each descent and each rebound of the ball, together with the velocity just before and just after each bounce. Recall that the time in seconds needed to fall h meters is $\sqrt{2h/g}$. The ball hits the ground with velocity $-g\sqrt{2h/g} = -\sqrt{2hg}$ (taking the upward direction to be positive) and rebounds with velocity $kg\sqrt{2h/g} = k\sqrt{2hg}$, taking time $k\sqrt{2h/g}$ to reach the top of its bounce, where its velocity is 0. At that point, its height is k^2h. All these results follow from the formulas for vertical motion with gravitational acceleration $-g$: $\frac{d^2y}{dt^2} = -g \;\Rightarrow\; v = \frac{dy}{dt} = v_0 - gt \;\Rightarrow\; y = y_0 + v_0t - \frac{1}{2}gt^2$.

number of descent	time of descent	speed before bounce	speed after bounce	time of ascent	peak height
1	$\sqrt{2H/g}$	$\sqrt{2Hg}$	$k\sqrt{2Hg}$	$k\sqrt{2H/g}$	k^2H
2	$\sqrt{2k^2H/g}$	$\sqrt{2k^2Hg}$	$k\sqrt{2k^2Hg}$	$k\sqrt{2k^2H/g}$	k^4H
3	$\sqrt{2k^4H/g}$	$\sqrt{2k^4Hg}$	$k\sqrt{2k^4Hg}$	$k\sqrt{2k^4H/g}$	k^6H
...	...	...	...	...	...

The total travel time in seconds is

$$\sqrt{\frac{2H}{g}} + k\sqrt{\frac{2H}{g}} + k\sqrt{\frac{2H}{g}} + k^2\sqrt{\frac{2H}{g}} + k^2\sqrt{\frac{2H}{g}} + \cdots = \sqrt{\frac{2H}{g}}\left(1 + 2k + 2k^2 + 2k^3 + \cdots\right)$$
$$= \sqrt{\frac{2H}{g}}\left[1 + 2k\left(1 + k + k^2 + \cdots\right)\right] = \sqrt{\frac{2H}{g}}\left[1 + 2k\left(\frac{1}{1-k}\right)\right] = \sqrt{\frac{2H}{g}}\frac{1+k}{1-k}$$

Another method: We could use part (b). At the top of the bounce, the height is $k^2h = rh$, so $\sqrt{r} = k$ and the result follows from part (b).

53. $\sum_{n=2}^{\infty}(1+c)^{-n}$ is a geometric series with $a = (1+c)^{-2}$ and $r = (1+c)^{-1}$, so the series converges when $\left|(1+c)^{-1}\right| < 1 \;\Leftrightarrow\; |1+c| > 1 \;\Leftrightarrow\; 1+c > 1$ or $1+c < -1 \;\Leftrightarrow\; c > 0$ or $c < -2$. We calculate the sum of the series and set it equal to 2: $\frac{(1+c)^{-2}}{1-(1+c)^{-1}} = 2 \;\Leftrightarrow\; \left(\frac{1}{1+c}\right)^2 = 2 - 2\left(\frac{1}{1+c}\right) \;\Leftrightarrow$

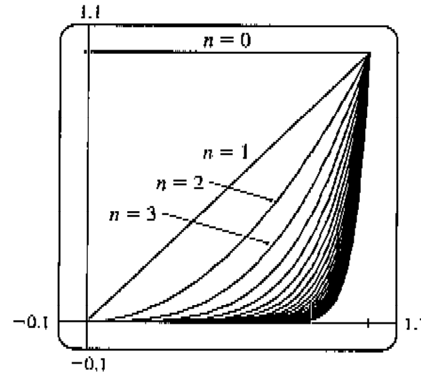
$1 = 2(1+c)^2 - 2(1+c) \;\Leftrightarrow\; 2c^2 + 2c - 1 = 0 \;\Leftrightarrow\; c = \frac{-2 \pm \sqrt{12}}{4} = \frac{\pm\sqrt{3}-1}{2}$. However, the negative root is inadmissible because $-2 < \frac{-\sqrt{3}-1}{2} < 0$. So $c = \frac{\sqrt{3}-1}{2}$.

54. The area between $y = x^{n-1}$ and $y = x^n$ for $0 \le x \le 1$ is

$$\int_0^1 \left(x^{n-1} - x^n\right) dx = \left[\frac{x^n}{n} - \frac{x^{n+1}}{n+1}\right]_0^1 = \frac{1}{n} - \frac{1}{n+1}$$

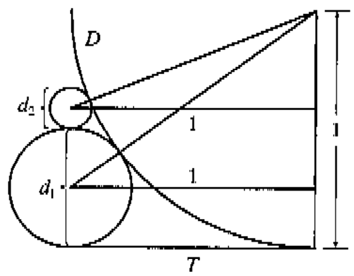
$$= \frac{(n+1)-n}{n(n+1)} = \frac{1}{n(n+1)}$$

We can see from the diagram that as $n \to \infty$, the sum of the areas between the successive curves approaches the area of the unit square, that is, 1. So $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.



55. Let d_n be the diameter of C_n. We draw lines from the centers of the C_i to the center of D (or C), and using the Pythagorean Theorem, we can write $1^2 + \left(1 - \frac{1}{2}d_1\right)^2 = \left(1 + \frac{1}{2}d_1\right)^2 \;\Leftrightarrow$
$1 = \left(1 + \frac{1}{2}d_1\right)^2 - \left(1 - \frac{1}{2}d_1\right)^2 = 2d_1$ (difference of squares)
$\Rightarrow \;\; d_1 = \frac{1}{2}$. Similarly,

$$1 = \left(1 + \tfrac{1}{2}d_2\right)^2 - \left(1 - d_1 - \tfrac{1}{2}d_2\right)^2 = 2d_2 + 2d_1 - d_1^2 - d_1 d_2$$
$$= (2 - d_1)(d_1 + d_2) \quad \Leftrightarrow$$



$d_2 = \dfrac{1}{2 - d_1} - d_1 = \dfrac{(1-d_1)^2}{2-d_1}$, $1 = \left(1 + \frac{1}{2}d_3\right)^2 - \left(1 - d_1 - d_2 - \frac{1}{2}d_3\right)^2 \;\Leftrightarrow\; d_3 = \dfrac{\left[1 - (d_1 + d_2)\right]^2}{2 - (d_1 + d_2)}$, and in general, $d_{n+1} = \dfrac{\left(1 - \sum_{i=1}^n d_i\right)^2}{2 - \sum_{i=1}^n d_i}$. If we actually calculate d_2 and d_3 from the formulas above, we find that they are $\frac{1}{6} = \frac{1}{2 \cdot 3}$ and $\frac{1}{12} = \frac{1}{3 \cdot 4}$ respectively, so we suspect that in general, $d_n = \frac{1}{n(n+1)}$. To prove this, we use induction: Assume that for all $k \le n$, $d_k = \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$. Then

$\sum_{i=1}^{n} d_i = 1 - \frac{1}{n+1} = \frac{n}{n+1}$ (telescoping sum). Substituting this into our formula for d_{n+1}, we get

$$d_{n+1} = \frac{\left[1 - \frac{n}{n+1}\right]^2}{2 - \left(\frac{n}{n+1}\right)} = \frac{\frac{1}{(n+1)^2}}{\frac{n+2}{n+1}} = \frac{1}{(n+1)(n+2)},$$ and the induction is complete.

Now, we observe that the partial sums $\sum_{i=1}^n d_i$ of the diameters of the circles approach 1 as $n \to \infty$; that is,
$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$, which is what we wanted to prove.

56. $|CD| = b\sin\theta$, $|DE| = |CD|\sin\theta = b\sin^2\theta$, $|EF| = |DE|\sin\theta = b\sin^3\theta$, ... Therefore,

$|CD| + |DE| + |EF| + |FG| + \cdots = b\sum_{n=1}^{\infty}\sin^n\theta = b\left(\dfrac{\sin\theta}{1-\sin\theta}\right)$ since this is a geometric series with

$r = \sin\theta$ and $|\sin\theta| < 1$ (because $0 < \theta < \frac{\pi}{2}$).

57. The series $1 - 1 + 1 - 1 + 1 - 1 + \cdots$ diverges (geometric series with $r = -1$) so we cannot say that $0 = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$.

58. If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$ by Theorem 6, so $\lim_{n\to\infty}\dfrac{1}{a_n} \neq 0$, and so $\sum_{n=1}^{\infty}\dfrac{1}{a_n}$ is divergent by the Test for Divergence.

59. $\sum_{n=1}^{\infty} ca_n = \lim_{n\to\infty}\sum_{i=1}^{n} ca_i = \lim_{n\to\infty} c\sum_{i=1}^{n} a_i = c\lim_{n\to\infty}\sum_{i=1}^{n} a_i = c\sum_{n=1}^{\infty} a_n$, which exists by hypothesis.

60. If $\sum ca_n$ were convergent, then $\sum(1/c)(ca_n) = \sum a_n$ would be also, by Theorem 8. But this is not the case, so $\sum ca_n$ must diverge.

61. Suppose on the contrary that $\sum(a_n + b_n)$ converges. Then $\sum(a_n + b_n)$ and $\sum a_n$ are convergent series. So by Theorem 8, $\sum[(a_n + b_n) - a_n]$ would also be convergent. But $\sum[(a_n + b_n) - a_n] = \sum b_n$, a contradiction, since $\sum b_n$ is given to be divergent.

62. No. For example, take $\sum a_n = \sum n$ and $\sum b_n = \sum(-n)$, which both diverge, yet $\sum(a_n + b_n) = \sum 0$, which converges with sum 0.

63. The partial sums $\{s_n\}$ form an increasing sequence, since $s_n - s_{n-1} = a_n > 0$ for all n. Also, the sequence $\{s_n\}$ is bounded since $s_n \leq 1000$ for all n. So by Theorem 12.1.11 [ET 11.1.11], the sequence of partial sums converges, that is, the series $\sum a_n$ is convergent.

64. (a) $$\text{RHS} = \frac{1}{f_{n-1}f_n} - \frac{1}{f_nf_{n+1}} = \frac{f_nf_{n+1} - f_nf_{n-1}}{f_n^2f_{n-1}f_{n+1}} = \frac{f_{n+1} - f_{n-1}}{f_nf_{n-1}f_{n+1}} = \frac{(f_{n-1} + f_n) - f_{n-1}}{f_nf_{n-1}f_{n+1}}$$
$$= \frac{1}{f_{n-1}f_{n+1}} = \text{LHS}$$

(b) $$\sum_{n=2}^{\infty}\frac{1}{f_{n-1}f_{n+1}} = \sum_{n=2}^{\infty}\left(\frac{1}{f_{n-1}f_n} - \frac{1}{f_nf_{n+1}}\right) \quad \text{[from part (a)]}$$
$$= \lim_{n\to\infty}\left[\left(\frac{1}{f_1f_2} - \frac{1}{f_2f_3}\right) + \left(\frac{1}{f_2f_3} - \frac{1}{f_3f_4}\right) + \left(\frac{1}{f_3f_4} - \frac{1}{f_4f_5}\right) + \cdots + \left(\frac{1}{f_{n-1}f_n} - \frac{1}{f_nf_{n+1}}\right)\right]$$
$$= \lim_{n\to\infty}\left(\frac{1}{f_1f_2} - \frac{1}{f_nf_{n+1}}\right) = \frac{1}{f_1f_2} - 0 = \frac{1}{1\cdot 1} = 1 \text{ because } f_n \to \infty \text{ as } n \to \infty.$$

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{f_n}{f_{n-1}f_{n+1}} = \sum_{n=2}^{\infty}\left(\frac{f_n}{f_{n-1}f_n} - \frac{f_n}{f_n f_{n+1}}\right)$ (as above)

$$= \sum_{n=2}^{\infty}\left(\frac{1}{f_{n-1}} - \frac{1}{f_{n+1}}\right)$$

$$= \lim_{n\to\infty}\left[\left(\frac{1}{f_1} - \frac{1}{f_3}\right) + \left(\frac{1}{f_2} - \frac{1}{f_4}\right) + \left(\frac{1}{f_3} - \frac{1}{f_5}\right) + \left(\frac{1}{f_4} - \frac{1}{f_6}\right) + \cdots + \left(\frac{1}{f_{n-1}} - \frac{1}{f_{n+1}}\right)\right]$$

$$= \lim_{n\to\infty}\left(\frac{1}{f_1} + \frac{1}{f_2} - \frac{1}{f_n} - \frac{1}{f_{n+1}}\right) = 1 + 1 - 0 - 0 = 2 \text{ because } f_n \to \infty \text{ as } n \to \infty.$$

65. (a) At the first step, only the interval $\left(\frac{1}{3}, \frac{2}{3}\right)$ (length $\frac{1}{3}$) is removed. At the second step, we remove the intervals $\left(\frac{1}{9}, \frac{2}{9}\right)$ and $\left(\frac{7}{9}, \frac{8}{9}\right)$, which have a total length of $2 \cdot \left(\frac{1}{3}\right)^2$. At the third step, we remove 2^2 intervals, each of length $\left(\frac{1}{3}\right)^3$. In general, at the nth step we remove 2^{n-1} intervals, each of length $\left(\frac{1}{3}\right)^n$, for a length of $2^{n-1} \cdot \left(\frac{1}{3}\right)^n = \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}$. Thus, the total length of all removed intervals is $\sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^{n-1} = \frac{1/3}{1 - 2/3} = 1$ (geometric series with $a = \frac{1}{3}$ and $r = \frac{2}{3}$). Notice that at the nth step, the leftmost interval that is removed is $\left(\left(\frac{1}{3}\right)^n, \left(\frac{2}{3}\right)^n\right)$, so we never remove 0, and 0 is in the Cantor set. Also, the rightmost interval removed is $\left(1 - \left(\frac{2}{3}\right)^n, 1 - \left(\frac{1}{3}\right)^n\right)$, so 1 is never removed. Some other numbers in the Cantor set are $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{9}$, $\frac{2}{9}$, $\frac{7}{9}$, and $\frac{8}{9}$.

(b) The area removed at the first step is $\frac{1}{9}$; at the second step, $8 \cdot \left(\frac{1}{9}\right)^2$; at the third step, $(8)^2 \cdot \left(\frac{1}{9}\right)^3$. In general, the area removed at the nth step is $(8)^{n-1}\left(\frac{1}{9}\right)^n = \frac{1}{9}\left(\frac{8}{9}\right)^{n-1}$, so the total area of all removed squares is

$$\sum_{n=1}^{\infty} \frac{1}{9}\left(\frac{8}{9}\right)^{n-1} = \frac{1/9}{1 - 8/9} = 1.$$

66. (a)

a_1	1	2	4	1	1	1000
a_2	2	3	1	4	1000	1
a_3	1.5	2.5	2.5	2.5	500.5	500.5
a_4	1.75	2.75	1.75	3.25	750.25	250.75
a_5	1.625	2.625	2.125	2.875	625.375	375.625
a_6	1.6875	2.6875	1.9375	3.0625	687.813	313.188
a_7	1.65625	2.65625	2.03125	2.96875	656.594	344.406
a_8	1.67188	2.67188	1.98438	3.01563	672.203	328.797
a_9	1.66406	2.66406	2.00781	2.99219	664.398	336.602
a_{10}	1.66797	2.66797	1.99609	3.00391	668.301	332.699
a_{11}	1.66602	2.66602	2.00195	2.99805	666.350	334.650
a_{12}	1.66699	2.66699	1.99902	3.00098	667.325	333.675

The limits seem to be $\frac{5}{3}$, $\frac{8}{3}$, 2, 3, 667, and 334. Note that the limits appear to be "weighted" more toward a_2. In general, we guess that the limit is $\dfrac{a_1 + 2a_2}{3}$.

(b) $a_{n+1} - a_n = \frac{1}{2}(a_n + a_{n-1}) - a_n = -\frac{1}{2}(a_n - a_{n-1}) = -\frac{1}{2}\left[\frac{1}{2}(a_{n-1} + a_{n-2}) - a_{n-1}\right]$

$$= -\tfrac{1}{2}\left[-\tfrac{1}{2}(a_{n-1} - a_{n-2})\right] = \cdots = \left(-\tfrac{1}{2}\right)^{n-1}(a_2 - a_1)$$

Note that we have used the formula $a_k = \frac{1}{2}(a_{k-1} + a_{k-2})$ a total of $n-1$ times in this calculation, once for each k between 3 and $n+1$. Now we can write

$$a_n = a_1 + (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_{n-1} - a_{n-2}) + (a_n - a_{n-1})$$
$$= a_1 + \textstyle\sum_{k=1}^{n-1}(a_{k+1} - a_k) = a_1 + \sum_{k=1}^{n-1}\left(-\frac{1}{2}\right)^{k-1}(a_2 - a_1)$$

and so

$$\lim_{n\to\infty} a_n = a_1 + (a_2 - a_1)\textstyle\sum_{k=1}^{\infty}\left(-\frac{1}{2}\right)^{k-1} = a_1 + (a_2 - a_1)\left[\frac{1}{1-(-1/2)}\right]$$
$$= a_1 + \tfrac{2}{3}(a_2 - a_1) = \frac{a_1 + 2a_2}{3}$$

67. (a) For $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$, $s_1 = \dfrac{1}{1\cdot 2} = \dfrac{1}{2}$, $s_2 = \dfrac{1}{2} + \dfrac{2}{1\cdot 2\cdot 3} = \dfrac{5}{6}$, $s_3 = \dfrac{5}{6} + \dfrac{3}{1\cdot 2\cdot 3\cdot 4} = \dfrac{23}{24}$,

$s_4 = \dfrac{23}{24} + \dfrac{4}{1\cdot 2\cdot 3\cdot 4\cdot 5} = \dfrac{119}{120}$. The denominators are $(n+1)!$, so a guess would be $s_n = \dfrac{(n+1)! - 1}{(n+1)!}$.

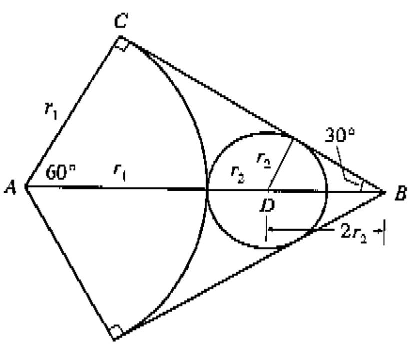
(b) For $n = 1$, $s_1 = \dfrac{1}{2} = \dfrac{2! - 1}{2!}$, so the formula holds for $n = 1$. Assume $s_k = \dfrac{(k+1)! - 1}{(k+1)!}$. Then

$$s_{k+1} = \frac{(k+1)! - 1}{(k+1)!} + \frac{k+1}{(k+2)!} = \frac{(k+1)! - 1}{(k+1)!} + \frac{k+1}{(k+1)!(k+2)}$$
$$= \frac{(k+2)! - (k+2) + k + 1}{(k+2)!} = \frac{(k+2)! - 1}{(k+2)!}$$

Thus, the formula is true for $n = k+1$. So by induction, the guess is correct.

(c) $\displaystyle\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{(n+1)! - 1}{(n+1)!} = \lim_{n\to\infty}\left[1 - \frac{1}{(n+1)!}\right] = 1$ and so $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1$.

68.



Let r_1 = radius of the large circle, r_2 = radius of next circle, and so on. From the figure we have $\angle BAC = 60°$ and $\cos 60° = r_1/|AB|$, so $|AB| = 2r_1$ and $|DB| = 2r_2$. Therefore, $2r_1 = r_1 + r_2 + 2r_2 = r_1 + 3r_2 \Rightarrow r_1 = 3r_2$. In general, we have $r_{n+1} = \frac{1}{3}r_n$, so the total area is

$$A = \pi r_1^2 + 3\pi r_2^2 + 3\pi r_3^2 + \cdots$$
$$= \pi r_1^2 + 3\pi r_2^2\left(1 + \frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{3^6} + \cdots\right)$$
$$= \pi r_1^2 + 3\pi r_2^2 \cdot \tfrac{1}{1-1/9} = \pi r_1^2 + \tfrac{27}{8}\pi r_2^2$$

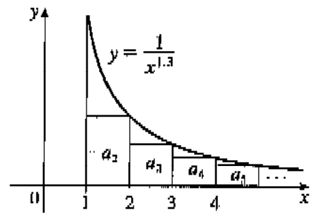
Since the sides of the triangle have length 1, $|BC| = \frac{1}{2}$ and $\tan 30° = \dfrac{r_1}{1/2}$. Thus, $r_1 = \dfrac{\tan 30°}{2} = \dfrac{1}{2\sqrt{3}} \Rightarrow$

$r_2 = \frac{1}{6\sqrt{3}}$, so $A = \pi\left(\frac{1}{2\sqrt{3}}\right)^2 + \frac{27\pi}{8}\left(\frac{1}{6\sqrt{3}}\right)^2 = \frac{\pi}{12} + \frac{\pi}{32} = \frac{11\pi}{96}$. The area of the triangle is $\frac{\sqrt{3}}{4}$, so the circles occupy about 83.1% of the area of the triangle.

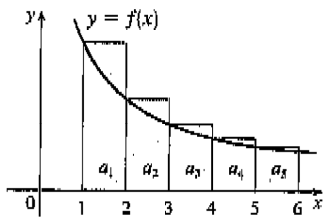
12.3 The Integral Test and Estimates of Sums ET 11.3

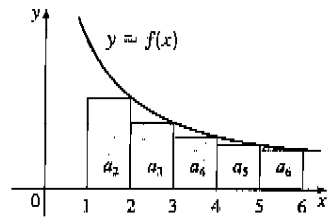
1. The picture shows that $a_2 = \dfrac{1}{2^{1.3}} < \displaystyle\int_1^2 \frac{1}{x^{1.3}}\,dx$,

$a_3 = \dfrac{1}{3^{1.3}} < \displaystyle\int_2^3 \frac{1}{x^{1.3}}\,dx$, and so on, so $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{1.3}} < \int_1^{\infty} \frac{1}{x^{1.3}}\,dx$. The integral converges by (8.8.2) [ET (7.8.2)] with $p = 1.3 > 1$, so the series converges.



2. From the first figure, we see that $\int_1^6 f(x)dx < \sum_{i=1}^5 a_i$. From the second figure, we see that $\sum_{i=2}^6 a_i < \int_1^6 f(x)\,dx$. Thus, we have $\sum_{i=2}^6 a_i < \int_1^6 f(x)\,dx < \sum_{i=1}^5 a_i$.





3. The function $f(x) = 1/x^4$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$\displaystyle\int_1^{\infty} \frac{1}{x^4}\,dx = \lim_{t\to\infty} \int_1^t x^{-4}\,dx = \lim_{t\to\infty} \left[\frac{x^{-3}}{-3}\right]_1^t = \lim_{t\to\infty}\left(-\frac{1}{3t^3} + \frac{1}{3}\right) = \frac{1}{3}$. Since this improper integral is convergent, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ is also convergent by the Integral Test.

4. The function $f(x) = 1/\sqrt[4]{x} = x^{-1/4}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$\int_1^{\infty} x^{-1/4}\,dx = \lim_{t\to\infty} \int_1^t x^{-1/4}\,dx = \lim_{t\to\infty} \left[\frac{4}{3}x^{3/4}\right]_1^t = \lim_{t\to\infty}\left(\frac{4}{3}t^{3/4} - \frac{4}{3}\right) = \infty$, so $\sum_{n=1}^{\infty} 1/\sqrt[4]{n}$ diverges.

5. The function $f(x) = 1/(3x+1)$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^{\infty} \frac{dx}{3x+1} = \lim_{b\to\infty} \int_1^b \frac{dx}{3x+1} = \lim_{b\to\infty} \left[\tfrac{1}{3}\ln(3x+1)\right]_1^b = \lim_{b\to\infty} \left[\tfrac{1}{3}\ln(3b+1) - \tfrac{1}{3}\ln 4\right] = \infty$$

so the improper integral diverges, and so does the series $\sum_{n=1}^{\infty} 1/(3n+1)$.

6. The function $f(x) = e^{-x}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$\int_1^{\infty} e^{-x}\,dx = \lim_{b\to\infty} \int_1^b e^{-x}\,dx = \lim_{b\to\infty} \left[-e^{-x}\right]_1^b = \lim_{b\to\infty} \left(-e^{-b} + e^{-1}\right) = e^{-1}$, so $\sum_{n=1}^{\infty} e^{-n}$ converges. *Note:* This is a geometric series, with first term $a = e^{-1}$ and ratio $r = e^{-1}$. Since $|r| < 1$, the series converges to $e^{-1}/(1 - e^{-1}) = 1/(e-1)$.

7. $f(x) = xe^{-x}$ is continuous and positive on $[1, \infty)$. $f'(x) = -xe^{-x} + e^{-x} = e^{-x}(1 - x) < 0$ for $x > 1$, so f is decreasing on $[1, \infty)$. Thus, the Integral Test applies.

$$\begin{aligned}\int_1^{\infty} xe^{-x}\,dx &= \lim_{b\to\infty} \int_1^b xe^{-x}\,dx = \lim_{b\to\infty} \left[-xe^{-x} - e^{-x}\right]_1^b \quad \text{(by parts)} \\ &= \lim_{b\to\infty} \left[-be^{-b} - e^{-b} + e^{-1} + e^{-1}\right] = 2/e\end{aligned}$$

since $\lim_{b\to\infty} be^{-b} = \lim_{b\to\infty} (b/e^b) \overset{H}{=} \lim_{b\to\infty} (1/e^b) = 0$ and $\lim_{b\to\infty} e^{-b} = 0$. Thus, $\sum_{n=1}^{\infty} ne^{-n}$ converges.