

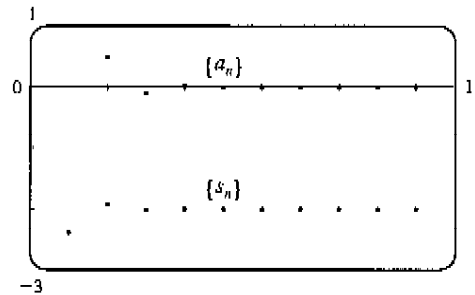
12.2 Series

ET 11.2

1. (a) A sequence is an ordered list of numbers whereas a series is the *sum* of a list of numbers.
- (b) A series is convergent if the sequence of partial sums is a convergent sequence. A series is divergent if it is not convergent.
2. $\sum_{n=1}^{\infty} a_n = 5$ means that by adding sufficiently many terms of the series we can get as close as we like to the number 5. In other words, it means that $\lim_{n \rightarrow \infty} s_n = 5$, where s_n is the n th partial sum, that is, $\sum_{i=1}^n a_i$.

3.

n	s_n
1	-2.40000
2	-1.92000
3	-2.01600
4	-1.99680
5	-2.00064
6	-1.99987
7	-2.00003
8	-1.99999
9	-2.00000
10	-2.00000



From the graph and the table, it seems that the series converges to -2 . In fact, it

is a geometric series with $a = -2.4$ and $r = -\frac{1}{5}$, so its sum is

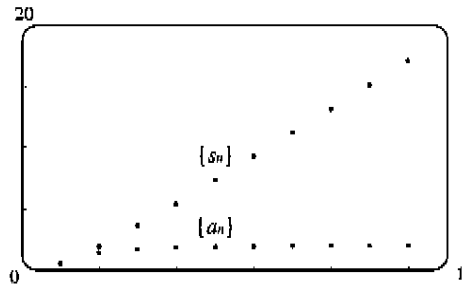
$$\sum_{n=1}^{\infty} \frac{12}{(-5)^n} = \frac{-2.4}{1 - (-\frac{1}{5})} = \frac{-2.4}{1.2} = -2. \text{ Note that the dot corresponding to}$$

$n = 1$ is part of both $\{a_n\}$ and $\{s_n\}$.

TI-86 Note: To graph $\{a_n\}$ and $\{s_n\}$, set your calculator to Param mode and DrawDot mode. (DrawDot is under GRAPH, MORE, FORMT (F3).) Now under E(t) = make the assignments: $x_{t1}=t$, $y_{t1}=12/(-5)^t$, $x_{t2}=t$, $y_{t2}=\text{sum seq}(y_{t1}, t, 1, t, 1)$. (sum and seq are under LIST, OPS (F5), MORE.) Under WIND use 1, 10, 1, 0, 10, 1, -3, 1, 1. to obtain a graph similar to the one above. Then use TRACE (F4) to see the values.

4.

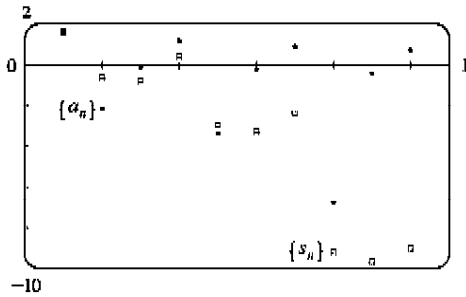
n	s_n
1	0.50000
2	1.90000
3	3.60000
4	5.42353
5	7.30814
6	9.22706
7	11.16706
8	13.12091
9	15.08432
10	17.05462



The series $\sum_{n=1}^{\infty} \frac{2n^2 - 1}{n^2 + 1}$ diverges, since its terms do not approach 0.

5.

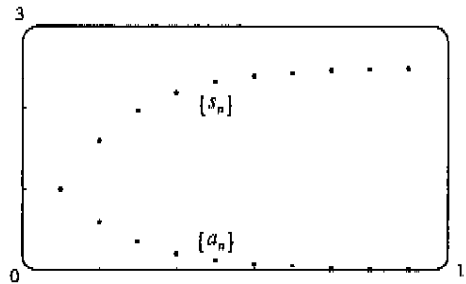
n	a_n
1	1.55741
2	-0.62763
3	-0.77018
4	0.38764
5	-2.99287
6	-3.28388
7	-2.41243
8	-9.21214
9	-9.66446
10	-9.01610



The series $\sum_{n=1}^{\infty} \tan n$ diverges, since its terms do not approach 0.

6.

n	a_n
1	1.00000
2	1.60000
3	1.96000
4	2.17600
5	2.30560
6	2.38336
7	2.43002
8	2.45801
9	2.47481
10	2.48488

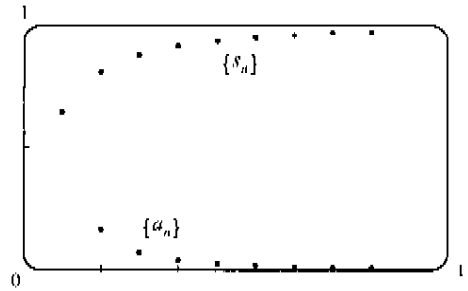


From the graph and the table, it seems that the series converges to 2.5. In fact, it is a geometric series with $a = 1$ and $r = 0.6$, so its sum is

$$\sum_{n=1}^{\infty} (0.6)^{n-1} = \frac{1}{1 - 0.6} = \frac{1}{2/5} = 2.5.$$

7.

n	s_n
1	0.64645
2	0.80755
3	0.87500
4	0.91056
5	0.93196
6	0.94601
7	0.95581
8	0.96296
9	0.96838
10	0.97259



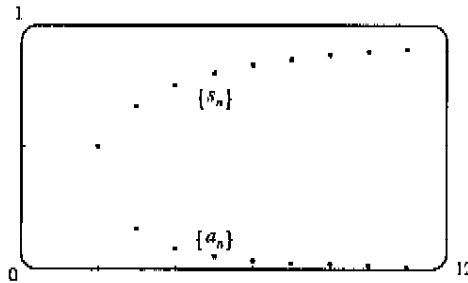
From the graph, it seems that the series converges to 1. To find the sum, we write

$$\begin{aligned}
 s_n &= \sum_{i=1}^n \left(\frac{1}{i^{1.5}} - \frac{1}{(i+1)^{1.5}} \right) \\
 &= \left(1 - \frac{1}{2^{1.5}} \right) + \left(\frac{1}{2^{1.5}} - \frac{1}{3^{1.5}} \right) + \left(\frac{1}{3^{1.5}} - \frac{1}{4^{1.5}} \right) + \cdots + \left(\frac{1}{n^{1.5}} - \frac{1}{(n+1)^{1.5}} \right) = 1 - \frac{1}{(n+1)^{1.5}}
 \end{aligned}$$

So the sum is $\lim_{n \rightarrow \infty} s_n = 1 - 0 = 1$.

8.

n	s_n
2	0.50000
3	0.66667
4	0.75000
5	0.80000
6	0.83333
7	0.85714
8	0.87500
9	0.88889
10	0.90000
11	0.90909
100	0.99000



From the graph and the table, it seems that the series converges to 1. To find the sum, we write

$$\begin{aligned}
 s_n &= \sum_{i=2}^n \frac{1}{i(i-1)} = \sum_{i=2}^n \left(\frac{1}{i-1} - \frac{1}{i} \right) \quad \text{[partial fractions]} \\
 &= \left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n} \right) = 1 - \frac{1}{n},
 \end{aligned}$$

and so the sum is $\lim_{n \rightarrow \infty} s_n = 1 - 0 = 1$.

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9. (a) $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{2n}{3n+1} = \frac{2}{3}$, so the *sequence* $\{a_n\}$ is convergent by (12.1.1) [ET (11.1.1)].
- (b) Since $\lim_{n \rightarrow \infty} a_n = \frac{2}{3} \neq 0$, the *series* $\sum_{n=1}^{\infty} a_n$ is divergent by the Test for Divergence (7).
10. (a) Both $\sum_{i=1}^n a_i$ and $\sum_{j=1}^n a_j$ represent the sum of the first n terms of the sequence $\{a_n\}$, that is, the n th partial sum.
- (b) $\sum_{i=1}^n a_j = \underbrace{a_j + a_j + \cdots + a_j}_{n \text{ terms}} = na_j$, which, in general, is not the same as $\sum_{i=1}^n a_i = a_1 + a_2 + \cdots + a_n$.
11. $3 + 2 + \frac{4}{3} + \frac{8}{9} + \cdots$ is a geometric series with first term $a = 3$ and common ratio $r = \frac{2}{3}$. Since $|r| = \frac{2}{3} < 1$, the series converges to $\frac{a}{1-r} = \frac{3}{1-2/3} = \frac{3}{1/3} = 9$.
12. $\frac{1}{8} - \frac{1}{4} + \frac{1}{2} - 1 + \cdots$ is a geometric series with $r = -2$. Since $|r| = 2 > 1$, the series diverges.
13. $-2 + \frac{5}{2} - \frac{25}{8} + \frac{125}{32} - \cdots$ is a geometric series with $a = -2$ and $r = \frac{5/2}{-2} = -\frac{5}{4}$. Since $|r| = \frac{5}{4} > 1$, the series diverges by (4).
14. $1 + 0.4 + 0.16 + 0.064 + \cdots$ is a geometric series with ratio 0.4. The series converges to $\frac{a}{1-r} = \frac{1}{1-2/5} = \frac{5}{3}$ since $|r| = \frac{2}{5} < 1$.
15. $\sum_{n=1}^{\infty} 5\left(\frac{2}{3}\right)^{n-1}$ is a geometric series with $a = 5$ and $r = \frac{2}{3}$. Since $|r| = \frac{2}{3} < 1$, the series converges to $\frac{a}{1-r} = \frac{5}{1-2/3} = \frac{5}{1/3} = 15$.
16. $\sum_{n=1}^{\infty} \frac{(-6)^{n-1}}{5^{n-1}}$ is a geometric series with $a = 1$ and $r = -\frac{6}{5}$. The series diverges since $|r| = \frac{6}{5} > 1$.
17. $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \frac{1}{4} \sum_{n=1}^{\infty} \left(-\frac{3}{4}\right)^{n-1}$. The latter series is geometric with $a = 1$ and $r = -\frac{3}{4}$. Since $|r| = \frac{3}{4} < 1$, it converges to $\frac{1}{1-(-3/4)} = \frac{4}{7}$. Thus, the given series converges to $\left(\frac{1}{4}\right)\left(\frac{4}{7}\right) = \frac{1}{7}$.
18. $\sum_{n=0}^{\infty} \frac{1}{(\sqrt{2})^n}$ is a geometric series with ratio $r = \frac{1}{\sqrt{2}}$. Since $|r| = \frac{1}{\sqrt{2}} < 1$, the series converges. Its sum is $\frac{1}{1-1/\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}-1} = \frac{\sqrt{2}}{\sqrt{2}-1} \cdot \frac{\sqrt{2}+1}{\sqrt{2}+1} = \sqrt{2}(\sqrt{2}+1) = 2 + \sqrt{2}$.
19. $\sum_{n=0}^{\infty} \frac{\pi^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{\pi}{3}\right)^n$ is a geometric series with ratio $r = \frac{\pi}{3}$. Since $|r| > 1$, the series diverges.
20. $\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}} = 3 \sum_{n=1}^{\infty} \left(\frac{e}{3}\right)^n$ is a geometric series with first term $3(e/3) = e$ and ratio $r = \frac{e}{3}$. Since $|r| < 1$, the series converges. Its sum is $\frac{e}{1-e/3} = \frac{3e}{3-e}$.
21. $\sum_{n=1}^{\infty} \frac{n}{n+5}$ diverges since $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n}{n+5} = 1 \neq 0$. [Use (7), the Test for Divergence.]

22. $\sum_{n=1}^{\infty} \frac{3}{n} = 3 \sum_{n=1}^{\infty} \frac{1}{n}$ diverges since each of its partial sums is 3 times the corresponding partial sum of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges. [If $\sum_{n=1}^{\infty} \frac{3}{n}$ were to converge, then $\sum_{n=1}^{\infty} \frac{1}{n}$ would also have to converge by Theorem 8(i).] In general, constant multiples of divergent series are divergent.

23. Using partial fractions, the partial sums are

$$\begin{aligned} s_n &= \sum_{i=2}^n \frac{2}{(i-1)(i+1)} = \sum_{i=2}^n \left(\frac{1}{i-1} - \frac{1}{i+1} \right) \\ &= \left(1 - \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \cdots + \left(\frac{1}{n-3} - \frac{1}{n-1} \right) + \left(\frac{1}{n-2} - \frac{1}{n} \right) \end{aligned}$$

This sum is a telescoping series and $s_n = 1 + \frac{1}{2} - \frac{1}{n-1} - \frac{1}{n}$.

$$\text{Thus, } \sum_{n=2}^{\infty} \frac{2}{n^2-1} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} - \frac{1}{n-1} - \frac{1}{n} \right) = \frac{3}{2}.$$

24. $\sum_{n=1}^{\infty} \frac{(n+1)^2}{n(n+2)}$ diverges by (7), the Test for Divergence, since

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n^2 + 2n + 1}{n^2 + 2n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n^2 + 2n} \right) = 1 \neq 0.$$

25. $\sum_{k=2}^{\infty} \frac{k^2}{k^2-1}$ diverges by the Test for Divergence since $\lim_{k \rightarrow \infty} a_k = \lim_{k \rightarrow \infty} \frac{k^2}{k^2-1} = 1 \neq 0$.

26. Converges. $s_n = \sum_{i=1}^n \frac{2}{i^2 + 4i + 3} = \sum_{i=1}^n \left(\frac{1}{i+1} - \frac{1}{i+3} \right)$ (using partial fractions). The latter sum is

$$\left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{4} - \frac{1}{6} \right) + \left(\frac{1}{5} - \frac{1}{7} \right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+2} \right) + \left(\frac{1}{n+1} - \frac{1}{n+3} \right) = \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}$$

$$\text{(telescoping series). Thus, } \sum_{n=1}^{\infty} \frac{2}{n^2 + 4n + 3} = \lim_{n \rightarrow \infty} \left(\frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3} \right) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.$$

27. Converges. $\sum_{n=1}^{\infty} \frac{3^n + 2^n}{6^n} = \sum_{n=1}^{\infty} \left(\frac{3^n}{6^n} + \frac{2^n}{6^n} \right) = \sum_{n=1}^{\infty} \left[\left(\frac{1}{2} \right)^n + \left(\frac{1}{3} \right)^n \right] = \frac{1/2}{1-1/2} + \frac{1/3}{1-1/3} = 1 + \frac{1}{2} = \frac{3}{2}$

28. $\sum_{n=1}^{\infty} [(0.8)^{n-1} - (0.3)^n] = \sum_{n=1}^{\infty} (0.8)^{n-1} - \sum_{n=1}^{\infty} (0.3)^n$ [difference of two convergent geometric series]
 $= \frac{1}{1-0.8} - \frac{0.3}{1-0.3} = 5 - \frac{3}{7} = \frac{32}{7}.$

29. $\sum_{n=1}^{\infty} \sqrt[n]{2} = 2 + \sqrt{2} + \sqrt[3]{2} + \sqrt[4]{2} + \cdots$ diverges by the Test for Divergence since

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \sqrt[n]{2} = \lim_{n \rightarrow \infty} 2^{1/n} = 2^0 = 1 \neq 0.$$

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30. $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \ln\left(\frac{n}{2n+5}\right) = \lim_{n \rightarrow \infty} \ln\left(\frac{1}{2+5/n}\right) = \ln \frac{1}{2} \neq 0$, so the series diverges by the Test for Divergence.

31. $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \arctan n = \frac{\pi}{2} \neq 0$, so the series diverges by the Test for Divergence.

32. $\sum_{k=1}^{\infty} (\cos 1)^k$ is a geometric series with ratio $r = \cos 1 \approx 0.540302$. It converges because $|r| < 1$. Its sum is $\frac{\cos 1}{1 - \cos 1} \approx 1.175343$.

33. The first series is a telescoping sum:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{3}{n(n+3)} &= \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+3} \right) = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+2} - \frac{1}{n+3} \right) \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) + \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right) + \sum_{n=1}^{\infty} \left(\frac{1}{n+2} - \frac{1}{n+3} \right) \\ &= 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6} \end{aligned}$$

The second series is geometric with first term $\frac{5}{4}$ and ratio $\frac{1}{4}$: $\sum_{n=1}^{\infty} \frac{5}{4^n} = \frac{5/4}{1-1/4} = \frac{5}{3}$. Thus,

$$\sum_{n=1}^{\infty} \left(\frac{3}{n(n+3)} + \frac{5}{4^n} \right) = \sum_{n=1}^{\infty} \frac{3}{n(n+3)} + \sum_{n=1}^{\infty} \frac{5}{4^n} \text{ [sum of two convergent series]} = \frac{11}{6} + \frac{5}{3} = \frac{7}{2}.$$

34. $\sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{2}{n} \right)$ diverges because $\sum_{n=1}^{\infty} \frac{2}{n} = 2 \sum_{n=1}^{\infty} \frac{1}{n}$ diverges. (If it converged, then $\frac{1}{2} \cdot 2 \sum_{n=1}^{\infty} \frac{1}{n}$ would also converge by Theorem 8(i), but we know from Example 7 that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.) If the given series converges, then the difference $\sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{2}{n} \right) - \sum_{n=1}^{\infty} \frac{3}{5^n}$ must converge (since $\sum_{n=1}^{\infty} \frac{3}{5^n}$ is a convergent geometric series) and equal $\sum_{n=1}^{\infty} \frac{2}{n}$, but we have just seen that $\sum_{n=1}^{\infty} \frac{2}{n}$ diverges, so the given series must also diverge.

35. $0.\overline{2} = \frac{2}{10} + \frac{2}{10^2} + \cdots$ is a geometric series with $a = \frac{2}{10}$ and $r = \frac{1}{10}$. It converges to $\frac{a}{1-r} = \frac{2/10}{1-1/10} = \frac{2}{9}$.

36. $0.\overline{73} = \frac{73}{10^2} + \frac{73}{10^4} + \cdots = \frac{73/10^2}{1-1/10^2} = \frac{73/100}{99/100} = \frac{73}{99}$

37. $3.\overline{417} = 3 + \frac{417}{10^3} + \frac{417}{10^6} + \cdots = 3 + \frac{417/10^3}{1-1/10^3} = 3 + \frac{417}{999} = \frac{3414}{999} = \frac{1138}{333}$

38. $6.\overline{254} = 6.2 + \frac{54}{10^3} + \frac{54}{10^6} + \cdots = 6.2 + \frac{54/10^3}{1-1/10^3} = \frac{62}{10} + \frac{54}{990} = \frac{6192}{990} = \frac{344}{55}$