KEY SECTION 10.2 HW PROBLEMS #8, 36, 38, 4

8) Find an equation of the tangent to the curve at the given point by two methods:

5pts

(a) without eliminating the parameter

(b) with eliminating the parameter

$x = \tan\theta$

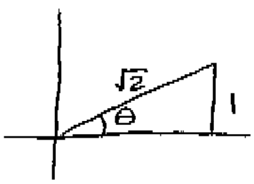
$y = \sec\theta$

$(1, \sqrt{2})$

(a) $x = \tan\theta \quad y = \sec\theta$

$dx = \sec^2\theta \quad dy = \sec\theta\tan\theta$

$$\frac{dy}{dx} = \frac{\sec\theta\tan\theta}{\sec^2\theta} = \frac{\tan\theta}{\sec\theta} = \frac{\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}} = \frac{\sin\theta}{\cos\theta}\cdot\frac{\cos\theta}{1} = \sin\theta$$



$\sin\theta = \frac{1}{\sqrt{2}}$

$\sin\theta = \frac{\sqrt{2}}{2}$

$\theta = \sin^{-1}\frac{\sqrt{2}}{2}$

$\theta = \frac{\pi}{4}$

$$\frac{dy}{dx} = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$ slope

$y - y_1 = m(x - x_1)$

$$\boxed{y - \sqrt{2} = \frac{\sqrt{2}}{2}(x-1) \quad \text{or} \quad y = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}}$$

(b) $x = \tan\theta \quad y = \sec\theta \quad (1, \sqrt{2})$

using: $\tan^2\theta + 1 = \sec^2\theta$

we get: $x^2 + 1 = y^2$

take derivative wrt y: $\frac{d}{dx}(x^2+1) = \frac{d}{dx}(y^2)$

$$2x = 2y\frac{dy}{dx}$$

$$\frac{2x}{2y} = \frac{dy}{dx}$$

$$\frac{x}{y} = \frac{dy}{dx}$$

$$\frac{1}{\sqrt{2}} = \frac{dy}{dx}$$

slope

$$\frac{\sqrt{2}}{2} = \frac{dy}{dx}$$

$$\boxed{y - \sqrt{2} = \frac{\sqrt{2}}{2}(x-1) \quad \text{or} \quad y = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}}$$

36) Let R be the region enclosed by the loop of the curve in Example 1

6pts (a) Find the area of R

(b) If R is rotated about the x-axis, find the volume of the resulting solid

(c) Find the centroid of R

(a) $x = t^2,\ y = t^3 - 3t,\ -\sqrt{3} \le t \le 0$

$dx = 2t\,dt$

$A = 2\int_0^{-\sqrt{3}} (t^3 - 3t)\,2t\,dt$

$A = 2\int_0^{-\sqrt{3}} (2t^4 - 6t^2)\,dt$

$A = 2\left[\frac{2}{5}t^5 - 2t^3\right]_0^{-\sqrt{3}}$

$A = 2\left[\frac{2}{5}(-\sqrt{3})^5 - 2(-\sqrt{3})^3\right]$

$A = 2\left[\frac{2}{5}(-9\sqrt{3}) - 2(-3\sqrt{3})\right]$

$A = 2\left[-\frac{18}{5}\sqrt{3} + 6\sqrt{3}\right]$

$\boxed{A = \frac{24}{5}\sqrt{3}}$

(b) $V = \pi\int_0^{-\sqrt{3}} (t^3 - 3t)^2\,2t\,dt$

$V = 2\pi\int_0^{-\sqrt{3}} (t^6 - 6t^4 + 9t^2)\,t\,dt$

$V = 2\pi\int_0^{-\sqrt{3}} (t^7 - 6t^5 + 9t^3)\,dt$

$V = 2\pi\left[\frac{1}{8}t^8 - t^6 + \frac{9}{4}t^4\right]_0^{-\sqrt{3}}$

$V = 2\pi\left[\frac{1}{8}(-\sqrt{3})^8 - (\sqrt{3})^6 + \frac{9}{4}(-\sqrt{3})^4\right]$

$V = 2\pi\left[\frac{81}{8} - 27 + \frac{81}{4}\right]$

$V = 2\pi\left[\frac{27}{8}\right]$

$\boxed{V = \frac{27}{4}\pi}$

(c) $\frac{24}{5}\sqrt{3} \cdot \frac{1}{2} = \frac{12}{5}\sqrt{3}$ reciprocal $= \frac{5}{12\sqrt{3}}$

$X = \frac{5}{12\sqrt{3}}\int_0^{-\sqrt{3}} t^2(t^3 - 3t)\,2t\,dt$

$x = \frac{5}{6\sqrt{3}}\int_0^{-\sqrt{3}} (t^6 - 3t^4)\,dt$

$x = \frac{5}{6\sqrt{3}}\left[\frac{1}{7}t^7 - \frac{3}{5}t^5\right]_0^{-\sqrt{3}}$

$x = \frac{5}{6\sqrt{3}}\left[\frac{1}{7}(-\sqrt{3})^7 - \frac{3}{5}(-\sqrt{3})^5\right]$

$x = \frac{5}{6\sqrt{3}}\left[-\frac{27}{7}\sqrt{3} + \frac{27}{5}\sqrt{3}\right]$

$x = \frac{9}{7}$ $\boxed{\text{coordinates of centroid are } \left(\frac{9}{7}, 0\right)}$

38) Set up, but do not evaluate, an integral that represents the length of the curve

4pts.

$$x = 1 + e^t \qquad y = t^2 \qquad -3 \le t \le 3$$

$$\frac{dx}{dt} = e^t \qquad \frac{dy}{dt} = 2t$$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$L = \int_{-3}^{3} \sqrt{(e^t)^2 + (2t)^2}\, dt$$

$$\boxed{L = \int_{-3}^{3} \sqrt{e^{2t} + 4t^2}\, dt}$$

46) Graph the curve and find its length

5pts.

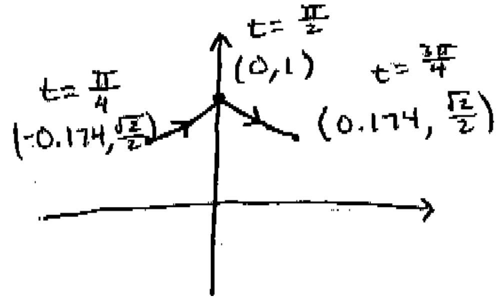
$$x = \cos t + \ln(\tan \tfrac{1}{2}t) \qquad y = \sin t \qquad \frac{\pi}{4} \le t \le \frac{3\pi}{4}$$

$$\frac{dx}{dt} = -\sin t + \frac{\frac{1}{2}\sec^2\left(\frac{t}{2}\right)}{\tan\left(\frac{t}{2}\right)} \qquad \frac{dy}{dt} = \cos t$$

$$= -\sin t + \frac{1}{2\cos^2\frac{t}{2}} \cdot \frac{\cos\frac{t}{2}}{\sin\frac{t}{2}}$$

$$= -\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}$$

$$= -\sin t + \frac{1}{\sin t}$$



$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(-\sin t + \frac{1}{\sin t}\right)^2 + (\cos t)^2$$

$$= \sin^2 t - 2 + \frac{1}{\sin^2 t} + \cos^2 t$$

$$= \sin^2 t + \cos^2 t - 2 + \csc^2 t$$

$$= 1 - 2 + \csc^2 t$$

$$= \csc^2 t - 1$$

$$= \cot^2 t$$

$$L = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} |\cot t|\, dt$$

$$L = 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot t\, dt$$

$$L = 2\Big[\ln|\sin t|\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$L = 2\left[\ln\left|\sin\tfrac{\pi}{2}\right| - \ln\left|\sin\tfrac{\pi}{4}\right|\right]$$

$$L = 2\left[\ln 1 - \ln\tfrac{\sqrt{2}}{2}\right]$$

$$L = 2\left[0 + \ln\tfrac{2}{\sqrt{2}}\right]$$

$$L = 2\left[\ln\sqrt{2}\right] = 2\left[\tfrac{1}{2}\ln 2\right]$$

$$L = \boxed{\ln 2}$$

72) (a) Show that the curvature at each point of a straight line is $k=0$

5pts (b) Show that the curvature at each point of a circle of radius r is $k=\frac{1}{r}$

$$K=\frac{|\dot{x}\ddot{y}-\ddot{x}\dot{y}|}{[\dot{x}^2+\dot{y}^2]^{3/2}}$$

(a) $x=a+vt$ $\quad y=b+wt$

$\dot{x}=v$ $\quad \dot{y}=w$

$\ddot{x}=0$ $\quad \ddot{y}=0$

$$K=\frac{|v\cdot 0-0\cdot w|}{[v^2+w^2]^{3/2}}$$

$$K=0$$

(b) $x=r\cos\theta$ $\quad y=r\sin\theta$

$\dot{x}=-r\sin\theta$ $\quad \dot{y}=r\cos\theta$

$\ddot{x}=-r\cos\theta$ $\quad \ddot{y}=-r\sin\theta$

$$K=\frac{|(-r\sin\theta)(-r\sin\theta)-(-r\cos\theta)(r\cos\theta)|}{[(-r\sin\theta)^2+(r\cos\theta)^2]^{3/2}}$$

$$K=\frac{|r^2\sin^2\theta+r^2\cos^2\theta|}{[r^2\sin^2\theta+r^2\cos^2\theta]^{3/2}}$$

$$K=\frac{|r^2(\sin^2\theta+\cos^2\theta)|}{[r^2(\sin^2\theta+\cos^2\theta)]^{3/2}}$$

$$K=\frac{|r^2|}{[r^2]^{3/2}}=\frac{r^2}{r^3}=\frac{1}{r}$$