13.4: Motion in Space: Velocity and Acceleration

If $\vec{r}(t)$ gives the position of a particle moving thru space, then (as expected) $\vec{v}(t) = \vec{r}\,'(t)$ and $\vec{a}(t) = \vec{r}\,''(t)$.

Ex1: Find and sketch r, v, a at $t = \pi/6$ if

$\vec{r}(t) = \langle \sin(t), 2\cos(t) \rangle$.

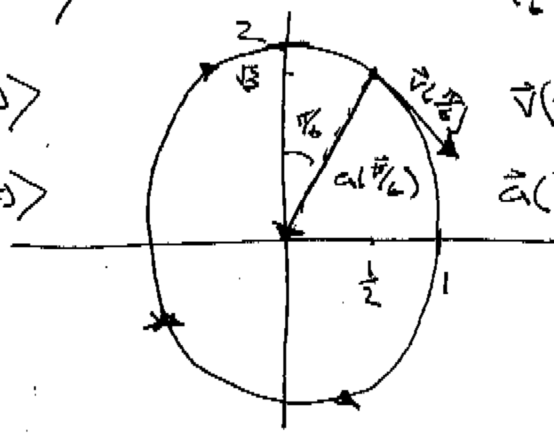
$\vec{r}(\frac{\pi}{6}) = \langle \frac{1}{2}, \sqrt{3} \rangle$

$\vec{v}(t) = \langle \cos(t), -2\sin(t) \rangle$

$\vec{v}(\frac{\pi}{6}) = \langle \frac{\sqrt{3}}{2}, -1 \rangle$

$\vec{a}(t) = \langle -\sin(t), -2\cos(t) \rangle$

$\vec{a}(\frac{\pi}{6}) = \langle -\frac{1}{2}, -\sqrt{3} \rangle$



Ex2: Find $\vec{v}(t)$ and $\vec{r}(t)$ if $\vec{a}(t) = \langle 0, 0, 1 \rangle$ and $\vec{v}(0) = \langle 1, -1, 0 \rangle$ and $r(0) = \vec{0}$.

Tangential and Normal components of Acceleration

$\vec{v}$ gives velocity and we use $v = |\vec{v}|$ to represent speed.

Now, recall $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \dfrac{\vec{v}(t)}{|\vec{v}(t)|} = \dfrac{\vec{v}}{v}$

$$\Rightarrow \vec{v} = v\vec{T} \Rightarrow \vec{v}\,' = \vec{a} = v'\vec{T} + v\vec{T}\,' \quad *$$

also recall $k = \dfrac{|\vec{T}\,'|}{|\vec{r}\,'|} = \dfrac{|\vec{T}\,'|}{v} \Rightarrow |\vec{T}\,'| = kv$.

but we defined the unit normal as $\vec{N} = \dfrac{\vec{T}\,'}{|\vec{T}\,'|}$

$$\Rightarrow \vec{T}\,' = \vec{N}|\vec{T}\,'| = kv\vec{N}$$

$*$ Hence $\vec{a} = v'\vec{T} + kv^2\vec{N}$

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Now we can write

$$\vec{a} = a_T \vec{T} + a_N \vec{N} \quad \text{where } a_T = v' \text{ and } a_N = k v^2$$

Find a_T and a_N in terms of $\vec{r}$.

a_T:
$$\vec{v} \cdot \vec{a} = v\vec{T} \cdot (v'\vec{T} + k v^2 \vec{N})$$
$$= v v' \vec{T} \cdot \vec{T} + k v^3 \vec{T} \cdot \vec{N}$$
$$= v \cdot v'$$
$$\Rightarrow a_T = v' = \frac{\vec{v} \cdot \vec{a}}{v} = \frac{\vec{r}'(t) \cdot \vec{r}''(t)}{|\vec{r}'(t)|}$$

a_N:
$$k v^2 = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3} |\vec{r}'(t)|^2 = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|}$$

Ex 3: A particle moves along the twisted cubic $\vec{r}(t) = \langle t, t^2, t^3 \rangle$. Find the tangential and normal components of acceleration.

13.4 Helix 1/2

Helix $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$

$\Rightarrow \vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$

AND $|\vec{r}\,'(t)| = \sqrt{2}$

SO $\vec{T}(t) = \left\langle \frac{-\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$ — TANGENT VECTOR

$\Rightarrow \vec{T}\,'(t) = \left\langle \frac{-\cos t}{\sqrt{2}}, \frac{-\sin t}{\sqrt{2}}, 0 \right\rangle$

AND $|\vec{T}\,'(t)| = \frac{1}{\sqrt{2}}$

SO $\vec{N}(t) = \langle -\cos t, -\sin t, 0 \rangle$ — NORMAL VECTOR

$$\Rightarrow \vec{B}(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{-\sin t}{\sqrt{2}} & \frac{\cos t}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\cos t & -\sin t & 0 \end{vmatrix} = \left\langle \frac{\sin t}{\sqrt{2}}, \frac{-\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

— BINORMAL VECTOR

To find the curvature

$$k = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|} = \frac{\frac{1}{\sqrt{2}}}{\sqrt{2}} = \frac{1}{2}$$ — curvature

(the radius of the kissing circle is $\frac{1}{k} = 2$)

The kissing circle

$$K(t,\theta) = \vec{r}(t) + \frac{1}{k(t)}\vec{N}(t) + \frac{1}{k(t)}\left(\vec{T}(t)\cos\theta + \vec{N}(t)\sin\theta\right)$$

$$= \langle \cos t, \sin t, t \rangle + \langle -2\cos t, -2\sin t, 0 \rangle$$

$$+ 2\left(\left\langle \frac{-\sin t\cos\theta}{\sqrt{2}}, \frac{\cos t\cos\theta}{\sqrt{2}}, \frac{\cos\theta}{\sqrt{2}} \right\rangle + \langle -\cos t\sin\theta, -\sin t\sin\theta, 0 \rangle \right)$$

$$= \left(-\cos t - \frac{2}{\sqrt{2}}\sin t\cos\theta - 2\cos t\sin\theta\right)\vec{i}$$

$$+ \left(-\sin t + \frac{2}{\sqrt{2}}\cos t\cos\theta - 2\sin t\sin\theta\right)\vec{j}$$

$$+ \left(t + \frac{2}{\sqrt{2}}\cos\theta\right)\vec{k}$$

the kissing circle

13.4
Helix
2/2

To find the tangential & normal components of acceleration...

$\vec{r}''(t) = \vec{a}(t) = \langle -\cos t, -\sin t, 0 \rangle$ — acceleration

$$a_T = \frac{\vec{r}'(t) \cdot \vec{r}''(t)}{|\vec{r}'(t)|}$$

$$= \frac{\langle -\sin t, \cos t, 1 \rangle \cdot \langle -\cos t, -\sin t, 0 \rangle}{\sqrt{2}}$$

$$= \frac{-2\sin t \cos t}{\sqrt{2}}$$ — tangential acceleration

$$a_N = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|}$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

$$= \frac{|\langle \sin t, -\cos t, 1 \rangle|}{\sqrt{2}}$$

$= 1$ — normal acceleration

So $\vec{a} = \frac{-2\sin t \cos t}{\sqrt{2}} \vec{T}(t) + 1\,\vec{N}(t)$