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$$\Rightarrow |\vec{r}\,'(t)| = \sqrt{a^2\sin^2 t + a^2\cos^2 t}$$
$$= \sqrt{a^2}$$
$$= |a|$$
$$= a \quad (\text{since } a > 0).$$

$$\Rightarrow \vec{T}(t) = \langle -\sin(t), \cos(t)\rangle$$
$$\Rightarrow \vec{T}\,'(t) = \langle -\cos(t), -\sin(t)\rangle$$

AND $|\vec{T}\,'(t)| = 1$.

So, $k(t) = \dfrac{1}{a}$ for $0 \le t \le 2\pi$.

Yet another curvature formula is:

$$k(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3} \qquad \boxed{3}$$

This formula is (perhaps) the easiest to apply.

And in the case where the curve is given by $y = f(x)$

$$k(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}} \qquad \boxed{4}$$

__Ex 4:__ Find the curvature of $\vec{r}(t) = \langle \cos(t), \sin(t), \cos(t)\rangle$

__Ex 5:__ Find the curvature of $y = \sin(x)$ on $\frac{\pi}{4} \le x \le \frac{\pi}{4}$.

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From your text (p 859), we have

$$* \quad \frac{d}{dt}\left[\vec{u}(t)\cdot\vec{v}(t)\right] = \vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}'(t)$$

Lemma: If $|\vec{r}(t)| = c$, then $\vec{r}(t)\cdot\vec{r}'(t) = 0$

proof.

$$|\vec{r}(t)|^2 = \vec{r}(t)\cdot\vec{r}(t)$$

$$\Rightarrow \frac{d}{dt}|\vec{r}(t)|^2 = \frac{d}{dt}\left[\vec{r}(t)\cdot\vec{r}(t)\right]$$

$$= 2\,\vec{r}(t)\cdot\vec{r}'(t)$$

$$= 0 \quad \left(\text{since } \frac{d}{dt}|\vec{r}(t)|^2 = \frac{d}{dt}c^2 = 0\right)$$

$\Rightarrow$ If $|\vec{r}(t)| = c$, then $\vec{r} \perp \vec{r}'$.

(3) Normal & BiNormal Vectors

Since $|\vec{T}(t)| = 1$ (a unit vector), we have $\vec{T}(t)\cdot\vec{T}'(t) = 0$ (by the lemma) & so $\vec{T} \perp \vec{T}'$.

Definition: The principle Normal Vector is

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$

Definition: The Binormal Vector is

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

Question: Why don't we ~~use~~ need to divide $\vec{B}$ by its magnitude?

Ex 6: Find $\vec{T}(t)$, $\vec{N}(t)$, and $\vec{B}(t)$ on

$$\vec{r}(t) = \langle 6\cos t, 8t, 6\sin t \rangle$$

1st: $\vec{r}\,'(t) = \langle -6\sin t, 8, 6\cos t \rangle$ and $|r'(t)| = 10$

so $\vec{T}(t) = \langle -\frac{6}{10}\sin t, \frac{8}{10}, \frac{6}{10}\cos t \rangle$

2nd: $\vec{T}\,'(t) = \langle -\frac{6}{10}\cos t, 0, -\frac{6}{10}\sin t \rangle$ and $|T'(t)| = \frac{6}{10}$

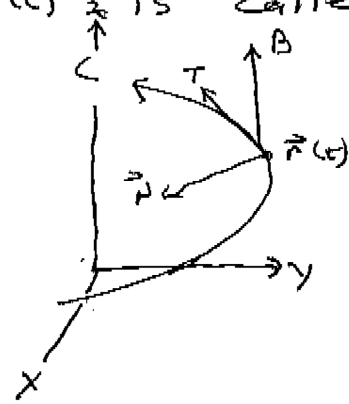
so $\vec{N}(t) = \langle -\cos t, 0, -\sin t \rangle$

3rd: $\vec{B}(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\frac{6}{10}\sin t & \frac{8}{10} & \frac{6}{10}\cos t \\ -\cos t & 0 & -\sin t \end{vmatrix}$

$$= \langle -\frac{8}{10}\sin t, -\frac{6}{10}, \frac{8}{10}\cos t \rangle$$

(4) kissing circle.

Definition: The plane determined by $\vec{B}$ and thru $\vec{r}(t)$ is called the osculating plane.



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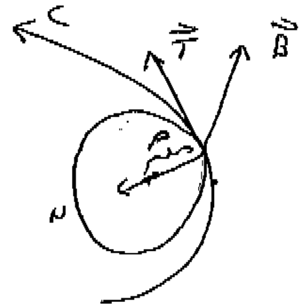
<u>Definition</u>: The osculating circle (or <u>kissing circle</u>) lies on the osculating plane, touches $\vec{r}(t)$, has radius $\rho = \frac{1}{k(t)}$ and has its center on the "side $\vec{N}$ is pointing to."



It shares
- a pt
- a tangent
- a plane
- curvature.

In 3 dimensions, the formula for the kissing circle is:

$$\mathbb{K}(t,\theta) = \underbrace{\vec{r}(t)}_{\text{starting pt on } \vec{r}(t)} + \underbrace{\frac{1}{k(t)} \cdot \vec{N}(t)}_{\substack{\text{move the circle's radius} \\ \text{in the direction of } \vec{N}(t)}} + \underbrace{\frac{1}{k(t)}}_{\substack{\text{radius of the} \\ \text{kissing circle}}} \underbrace{\left(\vec{T}(t)\cos\theta + \vec{N}(t)\sin(t)\right)}_{\substack{\text{a circle w/radius 1 on the plane} \\ \text{formed by } \vec{T} \text{ \& } \vec{N}.}}$$