13.3
1/e

13.3: Arclength and Curvature of Space Curves

Topics

- Arclength & Reparametrization.
- Curvature
- Normal & Binormal vectors
- The kissing Circle.

(1) Arclength

Recall from 10.2 that the arclength on $a \le t \le b$ of the curve expressed parametrically by $x = x(t)$; $y = y(t)$ is given by $L = \int_a^b \sqrt{\dot{x}^2 + \dot{y}^2}\, dt.$

This extends to finding the arclength of vector fct $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ as

$$L = \int_a^b \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt$$

$$= \int_a^b |\vec{r}\,'(t)|\, dt$$

assuming the curve is ~~traversed~~ only once on $a \le t \le b$.

133
2/8

Ex1: Find the length of $\vec{r}(t) = \langle 3\sin t, \overset{4}{\cancel{7}}t, -3\cos t \rangle$ on $-\pi \le t \le \pi$.

solution: $\vec{r}\,'(t) = \langle 3\cos t, \overset{4}{\cancel{7}}, 3\sin t \rangle$

$$\Rightarrow |\vec{r}\,'(t)| = \sqrt{9\cos^2 t + \overset{16}{\cancel{49}} + 9\sin^2 t}$$

$$\Rightarrow |\vec{r}\,'(t)| = \sqrt{9(\cos^2 t + \sin^2 t) + 16}$$

$$\Rightarrow |\vec{r}(t)| = \sqrt{25} = 5.$$

$$\text{AND} \quad L = \int_{-\pi}^{\pi} 5\,dt = 5t\Big|_{-\pi}^{\pi} \quad 10\pi$$

The parameter t can be troublesome because it has no direct relation to the curve itself. So, we want a new parameter s from the arclength from a given pt $\vec{r}_0$.

s says, "If you travel s units along the curve from $\vec{r}_0$, your new position will be $\vec{r}(s)$."

So, if C is a piecewise smooth (def. p859) curve given by $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ on $a \le t \le b$ and ~~at least one of x, y, z is 1-1~~ C is traversed exactly once on $a \le t \le b$, then

13.3
3/8

$$S(t) = \int_a^t |\vec{r}\,'(u)|\,du \quad \text{(dummy variable } u\text{)}$$

Differentiating WRT t

$$\frac{ds}{dt} = s'(t) = |\vec{r}\,'(t)|$$

Ex2: Reparametrize $\vec{r}(t) = \langle 3\sin t, 4t, -3\cos t \rangle$ WRT arclength measured from $(0,0,-3)$ in the direction of an increasing t.

Solution: we 1st need to find $s(t)$ so that we can the write $\vec{r}(s(t))$. or $\vec{r}(t(s))$ (possible since the curve was traversed once).

$$s(t) = \int_0^t |\vec{r}\,'(u)|\,du$$

$$= \int_0^t 5\,du$$

$$= 5t$$

AND $s = 5t \Rightarrow t = s/5$

$$\Rightarrow \vec{r}[t(s)] = \vec{r}(s) = \left\langle 3\sin\left(\frac{s}{5}\right), \frac{4s}{5}, -3\cos\left(\frac{s}{5}\right) \right\rangle$$

12.3
4/8

(2) Curvature

If C is a smooth space curve given by $\vec{r}(t)$, then $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$.

Q: When does $\vec{T}$ change quickly ... slowly

Def: The curvature of C is defined by

$$k = \left| \frac{dT}{ds} \right| \qquad \boxed{1}$$

where $\vec{T}$ is the unit tangent vector & s is the arclength.

To compute the curvature, we sometimes use ....

$$\frac{dT}{ds} = \frac{\frac{dT}{dt}}{\frac{ds}{dt}} = \frac{\vec{T}\,'(t)}{|\vec{r}\,'(t)|}$$

AND $$k = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|} \qquad \boxed{2}$$

Ex: Find the curvature of a circle w/ radius a.

solution: $\vec{r}(t) = \langle a\cos(t), a\sin(t) \rangle$ on $[0, 2\pi]$

$\Rightarrow \vec{r}\,'(t) = \langle -a\sin(t), a\cos(t) \rangle$

we need $\vec{T}\,'(t)$, so we need $\vec{T} = \dfrac{\vec{r}\,'}{|\vec{r}\,'|}$