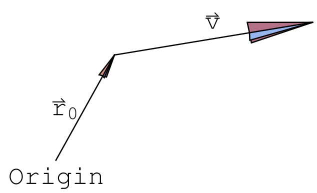
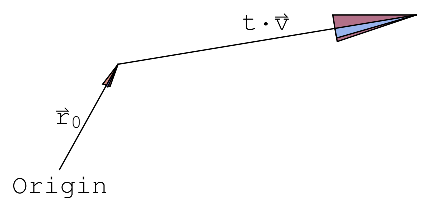
Math 126: Calculus III

Section 12.5: Equation of Lines and Planes

Vector equations of lines

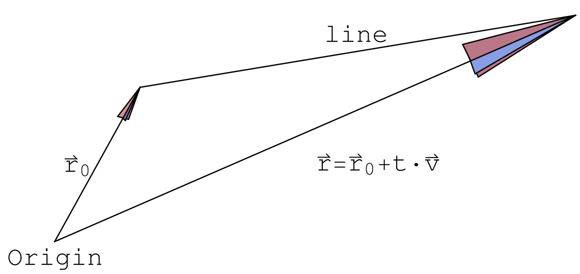
Consider the position vector $\vec{r}_0$ (which gives points to a point on the line) and the direction v
line). Additional points on the line can be reached by multiplying the vector $\vec{v}$ by the scaling





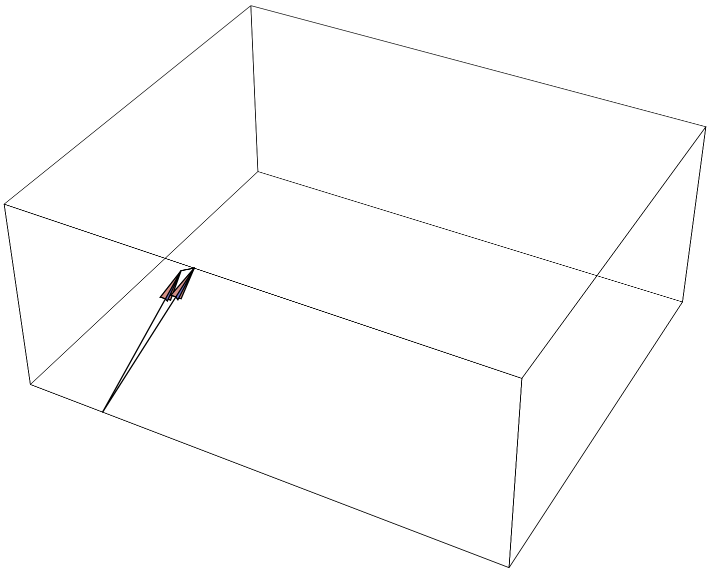
Vector equations of lines, continued

Thus, a line in $\mathbb{R}^n$ through a point given by the position vector $\vec{r}_0$ and in the direction of $\vec{v}$ ca
$\vec{r} = \vec{r}_0 + t \cdot \vec{v}$ (where t is a parameter).



Vector equations of lines, continued

So, when we allow t to vary, we are left with a line as below (animation)



Lines: Parametric Equations

So, if $\vec{r}_0$ is a position vector (for a point on the line), $\vec{v}$ is a direction vector for the line, and
(where t is a parameter), then:

$\vec{r}_0 = < x_0, y_0, z_0 >$
$\vec{v} = < a, b, c >$
and
$\vec{r} = < x, y, z >$

then $< x, y, z > = < x_0 + t\,a, y_0 + t\,b, z_0 + t\,c >$ is the vector equation of a line
parallel to the vector $\vec{v}$. Or, the parametric equations are: $x = x_0 + t\,a$; $y = y_0 + t\,b$; $z =$

Example 1

Find the parametric equations of the line through (1, 2, 3) and parallel to < 4, 5, 6 >.

Eliminating the Parameter

Eliminating the parameter leads to what are known as symmetric equations. That is, assumir
equation in

(1.) $x = x_0 + t\,a;\; y = y_0 + t\,b;\; z = z_0 + t\,c$ for the parameter t.

For example: $t = \frac{x - x_0}{a}$

Setting the three equations equal gives the symmetric equation

(2.) $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$ when $a, b, c \neq 0$

In the case where $\vec{v}$ has a zero component in a given direction (e.g., $a = 0$), then the symmet
$\frac{y - y_0}{b} = \frac{z - z_0}{c}$ when $b, c \neq 0$. This is a line on the plane $x = x_0$. (Note: these symmetric
(1.) rather than from the symmetric equations (2.)).

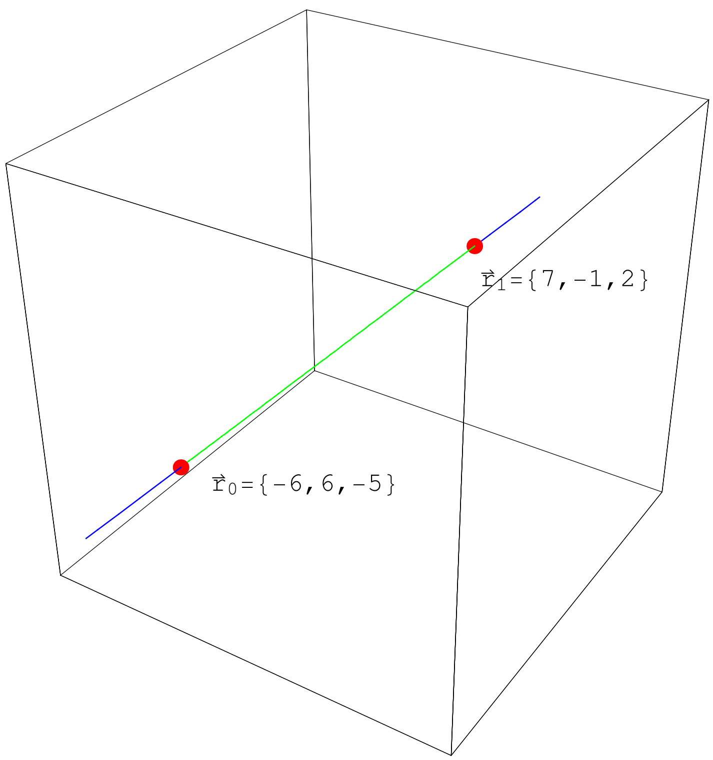
Example 2

a.) Find the symmetric equations of the line from example 1.

b.) Find the point were the line intersects the *xy* plane (where $z = 0$)

The Equation of Line between Points: The Picture

We are trying to find the line between two points. In the illustration, the points would be $\vec{r}_0$ =
green line represents the line segment between the two points (on $0 \le t \le 1$) and the blue line
for all values of t.



The Equation of Line between Points: The Formula

To describe the line from $\vec{r}_0$ to $\vec{r}_1$, begin with $1 \cdot \vec{r}_0 + 0 \cdot \vec{r}_1$ and go to $0 \cdot \vec{r}_0 + 1 \cdot \vec{r}_1$. T equation:

$$\vec{r} = (1 - t)\,\vec{r}_0 + t\,\vec{r}_1, \quad 0 \le t \le 1.$$

This is a powerful formula - understand the derivation rather than memorizing it.

Example 3

Find the parametric representation of a line from (1, 2, 3) to (3, 7, 11).

Planes in $\mathbb{R}^3$.

A plane can be determined by a point and a vector orthogonal to the plane (a normal vector).

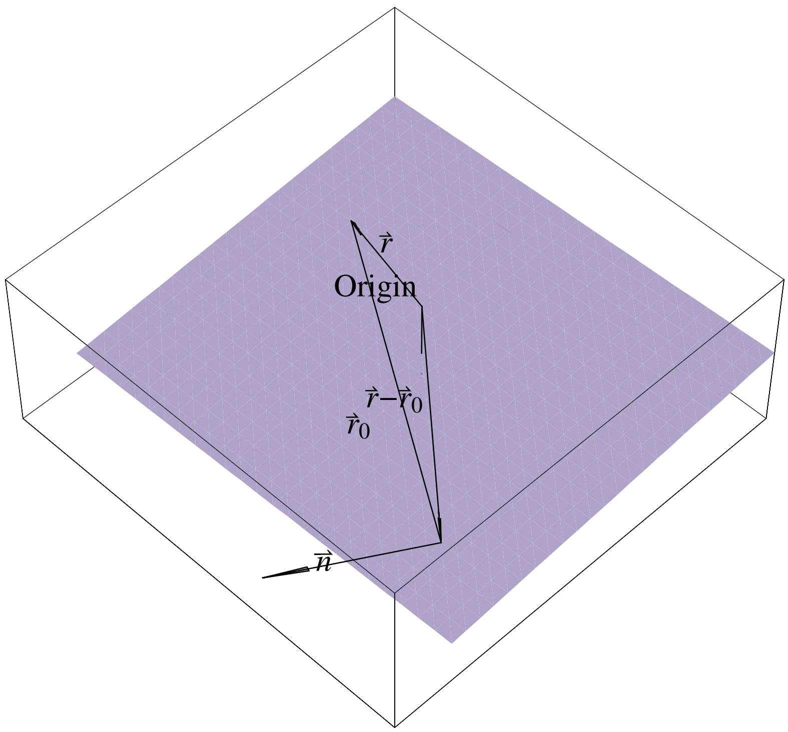
Why?

Notice that this is similar to finding a line by having its slope and a point.

1

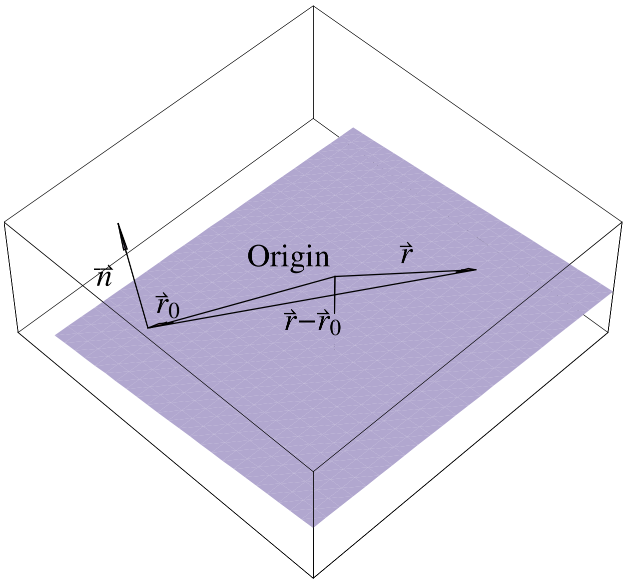
Forming a Plane - Seeing Dimension

The following animation shows two position vectors $\vec{r}_0$ and $\vec{r}$ of points on a plane, the vecto
a normal vector to the plane $\vec{n}$. The purpose of the animation is to help the student see the di



Forming a Plane - the Normal Vector

The following graph shows two position vectors $\vec{r}_0$ and $\vec{r}$ of points on a plane, the vector $\vec{r}$ - normal vector to the plane $\vec{n}$. The origin and z-axis are shown to help the student visualize d orthogonal to the vector $\vec{r} - \vec{r}_0$.



Thus, $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$.

The Equation of a Plane

Let $\vec{n}$, $\vec{r}$, and $\vec{r}_0$ be defined as follows:

$\vec{n} = < a, b, c >$ (normal vector)

$\vec{r} = < x, y, z >$ (position vector for an arbitrary point on the plane)

$\vec{r}_0 = < x_0, y_0, z_0 >$ (position vector for a given point on the plane)S

Since $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$, we have

$0 = < a, b, c > \cdot (< x, y, z > - < x_0, y_0, z_0 >)$

$0 = < a, b, c > \cdot < x - x_0, y - y_0, z - z_0 >$

$0 = a (x - x_0) + b (y - y_0) + c (z - z_0)$

This can also be written as $ax + by + cz + d = 0$

Example 4

Find the equation of the plane that includes the point `(1, 2, 3)` and that has normal vector

Question

How would we find the equation of the plane through three points (assuming that they are no

Definition

We define the angle between planes to be the angle between their normal vectors.

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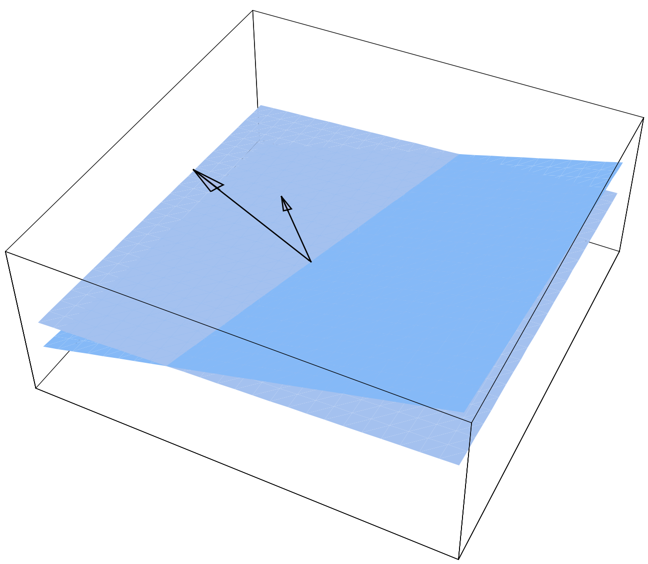
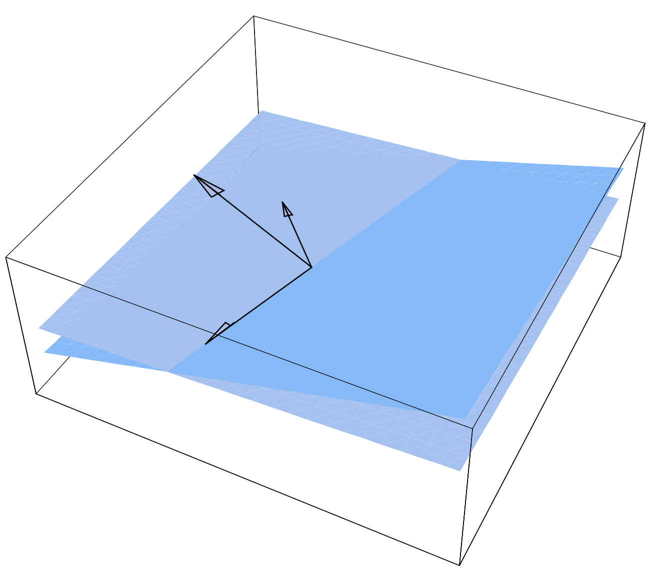
Example 5

Find the parametric equations for the line L of intersection of the planes z = x + y and 2 x -
Use the following steps.

1.) Find normal vectors to each plane.



2.) Find the cross product of the normals to determine the direction of the line L.



3.) Determine a point on the line `L` (one method would be to set `z = 0`).

In this case, when $z = 0$, we are left with $x = \frac{1}{3}$ and $y = -\frac{1}{3}$. Or, the line goes thru $\left(\frac{1}{3}, -\frac{1}{3},\right.$

4.) Express your answer.

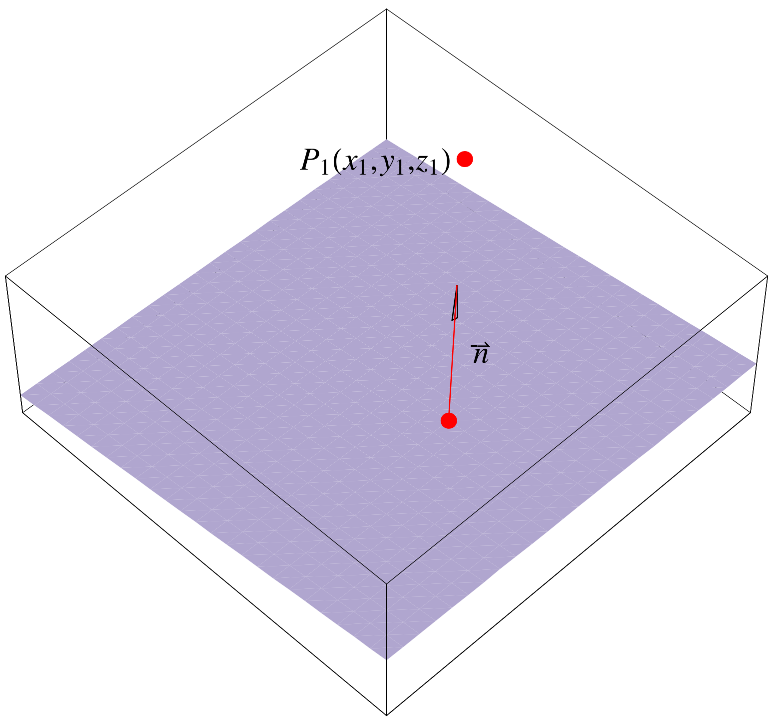
Note

There are an infinite number of vectors normal to a plane at a point.

Distance from a Plane to a Point - The Scenario

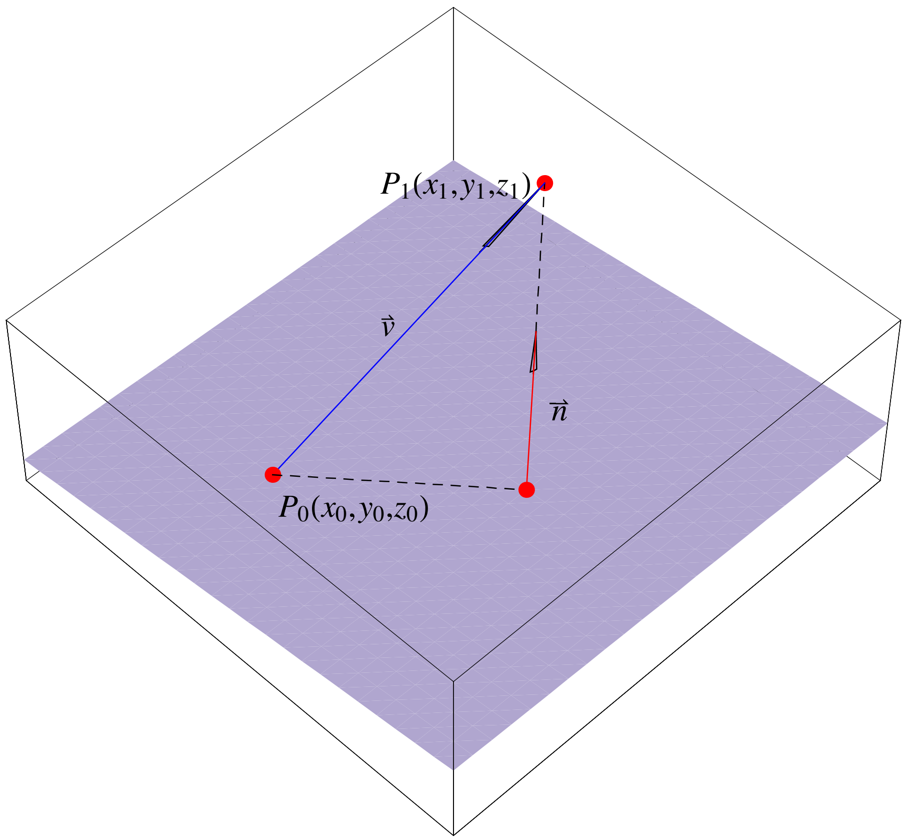
To find the distance D from a plane to a fixed point, we need to to find determine the magnit
plane and whose tip is at the point.

As we have seen, finding normal vectors to a plane is not difficult. Unfortunately, the norma
`ax + by + cz + d = 0` only gives the direction from the plane to the point, not the magnitud



Distance from a Plane to a Point - The Picture Deri

To find the distance D from the plane $ax + by + cz + d = 0$ to the point P_1, we will pick so
$ax_0 + by_0 + cz_0 + d = 0$) and find the magnitude of the projection of the vector $\vec{v}$ from P_0



Distance from a Plane to a Point - The Derivation

We have the following vectors:

$\vec{n} = \langle a, b, c \rangle$ (the normal to the plane $ax + by + cz + d = 0$)

$\vec{v} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$ (a vector from P_0 to P_1)

To find the distance D from the plane to P_1, we need to find:

$D = |\, \text{comp}_{\vec{n}} \vec{v} \,|$

$D = \frac{|\vec{n} \cdot \vec{v}|}{|\vec{n}|}$

$D = \frac{|a (x_1 - x_0) + b (y_1 - y_0) + c (z_1 - z_0)|}{|\vec{n}|}$

$D = \frac{|ax_1 - ax_0 + by_1 - by_0 + cz_1 - cz_0|}{|\vec{n}|}$

$D = \frac{|ax_1 - ax_0 + by_1 - by_0 + cz_1 - cz_0 + d - d|}{|\vec{n}|}$ (add $0 = d - d$)

$D = \frac{|ax_1 - ax_0 + by_1 - by_0 + cz_1 - cz_0 + d - d|}{|\vec{n}|}$ (recall that $ax_0 + by_0 + cz_0 + d = 0$ since P_0 is on

$D = \frac{|ax_1 + by_1 + cz_1 + d|}{|\vec{n}|}$

Mathematica Scratch Work: Lines in $\mathbb{R}^3$

```
Needs["VisualLA`"]
$TextStyle = {FontSize → 12}; $FormatType = TraditionalForm;
```

```
WINDOW := {{0, 10}, {0, 10}, {0, 10}};
Origin = {0, 0, 0};
A = {1, 2, 3};
B = {6, 5, 4};
V = B - A;
Show[{DrawVector3D[{A}, DisplayFunction → Identity],
   DrawVector3D[{V}, DisplayFunction → Identity]}, PlotRange → W
  DisplayFunction → $DisplayFunction];
```

```
Table[Show[DrawVector3D[{A + t V}, DisplayFunction → Identity],
   DisplayFunction → $DisplayFunction, PlotRange → WINDOW, Tick
  {t, 0, 1.5, 0.05}];
```

```
Table[Show[ParametricPlot3D[A + s B, {s, -0.05, t}, DisplayFunct
   DrawVector3D[{A + t B}, DisplayFunction → Identity], DisplayF
   PlotRange → WINDOW, Ticks → False], {t, 0, 1.5, 0.05}];
```

```
Table[Show[Graphics3D[Line[{A, A + t V}]], DrawVector3D[{A}, Di
   DrawVector3D[{A + t V}, DisplayFunction → Identity], DisplayF
   PlotRange → WINDOW, Ticks → False], {t, 0.05, 1.5, 0.05}];
```

```
WINDOW := {{-2, 10}, {0, 10}, {0, 5}};
```

```
t = 1.5;
P1 :=
  Show[
   {Graphics3D[{Text["Origin", {0, 0, 0}, {0, 1}], Text["r⃗0", A/2
      Text["v⃗", A + V/2, {0, -1}]}], DrawVector3D[{A}, DisplayFun
    DrawVector3D[{{A, A + V}}, DisplayFunction → Identity]}, Pl
   Ticks → False, Boxed → False];
P2 :=
  Show[
   {Graphics3D[{Text["Origin", {0, 0, 0}, {0, 1}], Text["r⃗0", A/2
      Text["t·v⃗", A + 1.5 V/2, {0, -1}]}], DrawVector3D[{A}, Disp
    DrawVector3D[{{A, A + t V}}, DisplayFunction → Identity]}, P
   Ticks → False, Boxed → False];
Show[GraphicsArray[{P1, P2}], DisplayFunction → $DisplayFuncti

P3 :=
  Show[
   {Graphics3D[{Text["Origin", {0, 0, 0}, {0, 1}], Text["r⃗0", A/2
      Text["line", A + 1.5 V/2, {0, -1}], Text["r⃗=r⃗0+t·v⃗", (A + t V)/2
    DrawVector3D[{A}, DisplayFunction → Identity],
    DrawVector3D[{A + t V}, DisplayFunction → Identity]}, PlotRa
   Ticks → False, Boxed → False];
P3;
```

```
WINDOW := {{-3, 3}, {-4, 3}, {-40, 20}};
f[x_, y_] := x + 2 y - 20;
P4 := Plot3D[f[x, y], {x, -3, 3}, {y, -3, 3}, Mesh → False, Displa
A = {2, -2, f[2, -2]};
B = {-2, 1, f[-2, 1]};
NORMAL = {-1, -2, +1};
t = 1;
Table[
  Show[
   {P4, Graphics3D[{Line[{A, B}], Line[{{0, 0, -40}, {0, 0, 0}}]
      Text["Origin", {0, 0, 0}, {1, -1}], Text["r⃗0", A/2, {4, 0}],
      Text["r⃗-r⃗0", (A + B)/2, {0, 1}], Text["n⃗", A + NORMAL/2, {2, 0}]
    DrawVector3D[{A}, HeadLength → 0.1, HeadAngle → 0.05, Displa
    DrawVector3D[{B}, HeadLength → 0.1, HeadAngle → 0.05, Displa
    DrawVector3D[{{A, A + NORMAL}}, HeadLength → 0.02, HeadAngle
     DisplayFunction → Identity]}, PlotRange → WINDOW, Ticks → F
   ViewPoint -> {3,  s, 5}, DisplayFunction → $DisplayFunction],


WINDOW := {{0, 4}, {-8, 1}, {-40, 20}};
f[x_, y_] := x + 2 y - 20;
P4 := Plot3D[f[x, y], {x, 0, 4}, {y, -8, 1}, Mesh → False, Display
A = {2, -2, f[2, -2]};
B = {3, -5, f[3, -5]};
NORMAL = {-1, -2, 1};
t = 1;
Show[{P4, PointPlot3D[{A, A + 2 NORMAL}, DisplayFunction → Identi
   Graphics3D[{Text["P1(x1,y1,z1)", A + 2 NORMAL, {1.1, 0}], Text
   DrawVector3D[{{A, A + NORMAL}}, HeadLength → 0.02, HeadAngle -
    DisplayFunction → Identity]}, PlotRange → WINDOW, Ticks → Fa
  ViewPoint -> {3, 3, 5}, DisplayFunction → $DisplayFunction];
```

```
WINDOW := {{0, 4}, {-8, 1}, {-40, 20}};
f[x_, y_] := x + 2 y - 20;
P4 := Plot3D[f[x, y], {x, 0, 4}, {y, -8, 1}, Mesh → False, Display
A = {2, -2, f[2, -2]};
B = {3, -5, f[3, -5]};
NORMAL = {-1, -2, 1};
t = 1;
Show[{P4, PointPlot3D[{A, A + 2 NORMAL, B}, DisplayFunction → Ide
   Graphics3D[{{Dashing[{0.01, 0.01}], Line[{A, B}], Line[{A + N
     Text["P0(x0,y0,z0)", B, {-1, 1.5}], Text["P1(x1,y1,z1)", A +
     Text["n⃗", A + NORMAL/2, {-2, 0}],
     Text["v⃗", (B + A + 2 NORMAL)/2, {2, 0}]}],
   DrawVector3D[{{B, A + 2 NORMAL}}, HeadLength → 0.05, HeadAngle
    DisplayFunction → Identity], DrawVector3D[{{A, A + NORMAL}},
    HeadAngle → 0.05, ShaftColor -> Red, DisplayFunction → Ident
  Ticks → False, Boxed → True, ViewPoint -> {3, 3, 5}, DisplayFunc
```

Line Segment between two points

In[288]:=
```
Needs["VisualLA`"]
r[t_] = (1 - t) {-6, 6, -5} + t {7, -1, 2};

A0 = ParametricPlot3D[Flatten[{r[t], RGBColor[0, 0, 1]}], {t, -5
   DisplayFunction -> Identity];
A1 = ParametricPlot3D[Flatten[{r[t], RGBColor[0, 0, 1]}], {t, 1,
   DisplayFunction -> Identity];
B = ParametricPlot3D[Flatten[{r[t], RGBColor[0, 1, 0]}], {t, 0, 1
PTS = PointPlot3D[{r[0], r[1]}, DisplayFunction → Identity];
LBL = {Text["r⃗0={-6,6,-5}", r[0], {-1.25, 1}], Text["r⃗1={7,-1,2}

Show[{Graphics3D[LBL], A0, B, A1, PTS}, DisplayFunction → $Displ
  PlotRange → {{-10, 10}, {-10, 10}, {-10, 10}}, Ticks → False, A
  Boxed → True, ViewPoint -> {-2.429, -1.548, 1.775}];
```

Intersection of planes

In[414]:=
```
(*Planes*)
P1[x_, y_] = x + y;
P2[x_, y_] = 2 x + 5 y + 1;
F = Plot3D[P1[x, y], {x, -3, 3}, {y, -3, 3}, DisplayFunction → Ide
G = Plot3D[P2[x, y], {x, -3, 3}, {y, -3, 3}, DisplayFunction → Ide

(*Vectors*)
N1 = {-1, -1, 1};
N2 = {-2, -5, 1};
N1xN2 = Cross[N1, N2];
PN1 = DrawVector3D[{{{1 / 3, -1 / 3, P1[1 / 3, -1 / 3]}, {1 / 3, -1 / 3,
   DisplayFunction → Identity];
PN2 = DrawVector3D[{{{1 / 3, -1 / 3, P2[1 / 3, -1 / 3]}, {1 / 3, -1 / 3,
   DisplayFunction → Identity];
DIRVEC = DrawVector3D[{{{1 / 3, -1 / 3, P2[1 / 3, -1 / 3]}, {1 / 3, -1
   DisplayFunction → Identity];

(*Graph*)
Show[{F, G, PN1, PN2}, DisplayFunction → $DisplayFunction, Ticks
Show[{F, G, PN1, PN2, DIRVEC}, DisplayFunction → $DisplayFunctic
  ViewPoint -> {2, 1, 2}];
```