FEB-04-2005 FRI 05:37 PM HCC MATH & BUSINESS FAX NO. 206 870 4850 P. 01/04

12.4
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12.4 : Cross Product

Determinants

Ex 1: Is there a unique solution to the system $\begin{cases} x + 2y = c_1 \\ 3x + 4y = c_2 \end{cases}$

To answer this, calculate the determinate $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1(4) - 2(3)$.

If the determinate is not zero then a unique solution exists.

Ex 2: Is there a unique solution to the system $\begin{cases} x + 2y + 3z = c_1 \\ 4x + 5y + 6z = c_2 \\ 7x + 8y + 9z = c_3 \end{cases}$

calculate the determinate

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1(45-48) - 2(36-42) + 3(32-35)$$
$$= -3 + 12 - 9$$
$$= 0$$

So, there is not a unique solution.

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Defn: Cross Product

If $\vec{u} = \langle u_1, u_2, u_3 \rangle$ & $\vec{v} = \langle v_1, v_2, v_3 \rangle$

then $u \times v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

ex 3 If $\vec{u} = \langle 1, 2, 3 \rangle$ and $\vec{v} = \langle 4, 5, 6 \rangle$, find $\vec{u} \times \vec{v}$
see page 5

ex4: Find $\vec{u} \overset{\text{DOT}}{\bullet} (\vec{u} \times \vec{v})$ w/ the vectors from (ex3)
see page 5

ex5: Find $\vec{u} \times \vec{u}$. (sol: $\vec{u} \times \vec{u} = \vec{0}$)

Thm: $\vec{u} \times \vec{v}$ is $\perp$ to $\vec{u}$ & $\vec{v}$.

Thm: $\vec{u} \times \vec{u} = \vec{0} \quad \forall u \in V_3$.

so, the cross product gives a vector $\vec{u} \times \vec{v}$ that is orthogonal to $\vec{u}$ & $\vec{v}$.

Geometrically, there are two directions that could be taken by $\vec{u} \times \vec{v} \longrightarrow$ the right hand rule.

w/ the ~~magnitude~~ direction in hand (the right one), what is the magnitude of $\vec{u} \times \vec{v}$?

Thm. If θ is the angle between $\vec{u}$ & $\vec{v}$ $(0 \leq \theta \leq \pi)$, then $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$

(lame uninviteful proof).

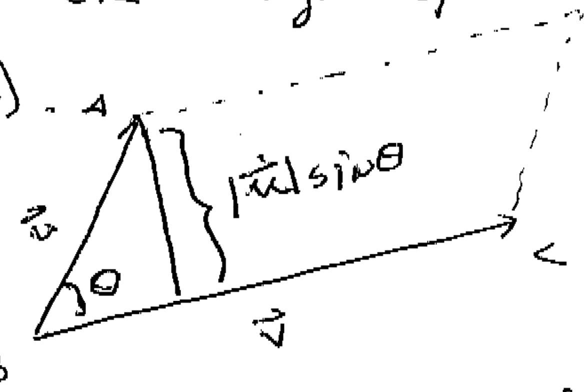
<u>Corollary</u>: Two non-zero vectors are // iff $\vec{u} \times \vec{v} = \vec{0}$ (why)

<u>ex6</u>: Find a vector $\perp$ to the plane that includes $A(1,2,3)$; $B(4,5,6)$; $C(9,8,7)$

$C(\cancel{7,8,9})$

see page 5

<u>ex7</u>: Find the area of the triangle w/ vertices A, B, C from (ex6).



$$\frac{|\vec{u}||\vec{v}|\sin\theta}{2} = \text{Area of } \triangle ABC$$

see page 6

<u>NOTE</u>: The $|\vec{u} \times \vec{v}|$ is maxed when $\vec{u} \perp \vec{v}$.

<u>NOTE</u>: Find $\vec{i} \times \vec{j}$, $\vec{i} \times \vec{k}$, $\vec{j} \times \vec{i}$, $\vec{k} \times \vec{i}$, etc.

<u>CROSS PRODUCT PROPERTIES</u> If $\vec{u}, \vec{v}, \vec{w} \in V_3$ & $c \in \mathbb{R}$

(1) $u \times v = -v \times u$

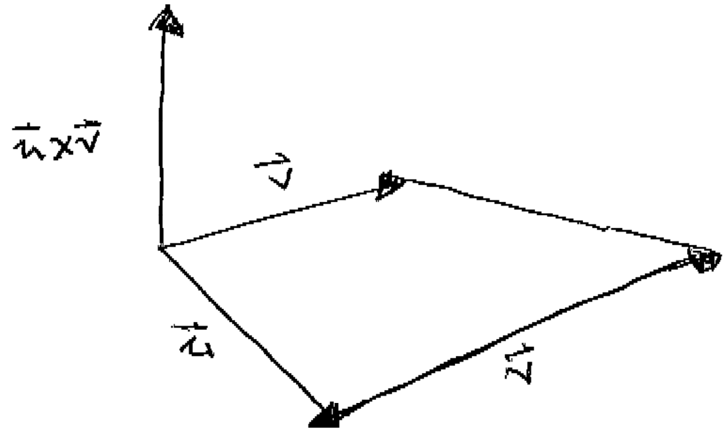
(2) $(c\,u) \times v = c(u \times v)$

(3) $u \times (v + w) = u \times v + u \times w$

(4) $(u + v) \times w = u \times w + v \times w$

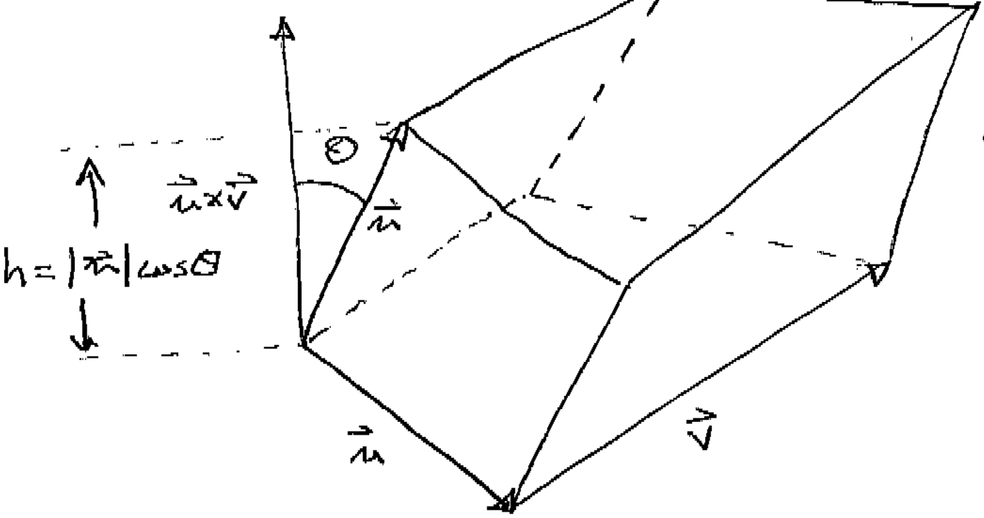
(5) $u \cdot (v \times w) = (u \times v) \cdot w$
(scalar triple product)

(6) $u \times (v \times w) = (u \cdot w)\, v - (u \cdot v)\, w$



$A = |\vec{u} \times \vec{v}|$

parallelogram



$$V = A \cdot h$$
$$= |\vec{v} \times \vec{w}|\,|\vec{u}| \cos\theta$$
$$= |\vec{u}|\,|\vec{v} \times \vec{w}| \cos\theta$$
$$= |\vec{u} \cdot (\vec{v} \times \vec{w})|$$

parallelpiped

scalar triple product

$$|\vec{u} \cdot (\vec{v} \times \vec{w})| = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

coplaner vectors: $|\vec{u} \cdot (\vec{v} \times \vec{w})| = 0$ (why?)

Torque: $\vec{\tau} = \vec{r} \times \vec{F}$ ($\vec{r}$ position)
($\vec{F}$ force)

ex3 work

$$\begin{vmatrix} \vec{i} & \vec{j} & k \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}\vec{k}$$

$$\left.\begin{aligned} &= -3\vec{i} + 6\vec{j} - 3\vec{k} \\ &= \langle -3, 6, -3 \rangle \end{aligned}\right\} \text{either ok}$$

ex4 work:

$$\vec{u} \cdot (\vec{u} \times \vec{v}) = \langle 1, 2, 3 \rangle \cdot \langle -3, 6, -3 \rangle$$

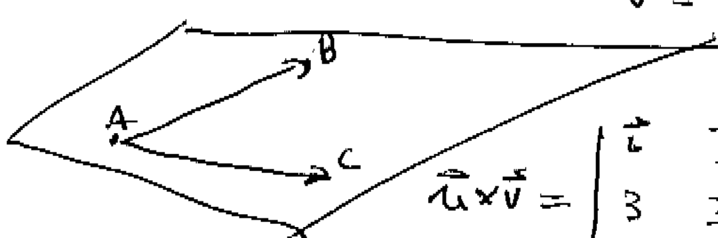
$$= -3 + 12 - 9$$

$$= 0 \quad (\text{so, } \vec{u} \text{ \& } (\vec{u} \times \vec{v}) \text{ are orthogonal})$$

ex 6 work:

$$\vec{u} = B - A = \langle 3, 3, 3 \rangle$$

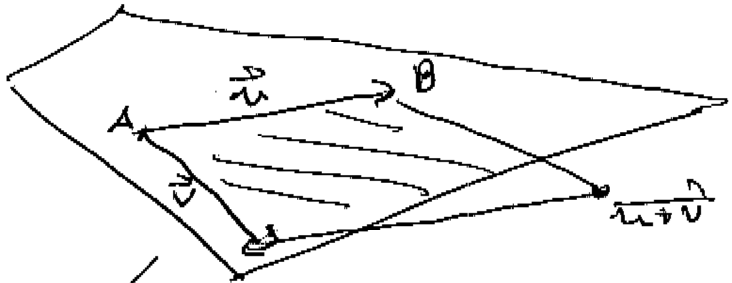
$$\vec{v} = C - A = \langle 8, 6, 4 \rangle$$



$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 3 & 3 \\ 8 & 6 & 4 \end{vmatrix} = \begin{vmatrix} 3 & 3 \\ 6 & 4 \end{vmatrix}\vec{i} - \begin{vmatrix} 3 & 3 \\ 8 & 4 \end{vmatrix}\vec{j} + \begin{vmatrix} 3 & 3 \\ 8 & 6 \end{vmatrix}\vec{k}$$

$$= -6\vec{i} + 12\vec{j} - 6\vec{j}$$

$= \langle -6, 12, -6 \rangle$ is a vec. $\perp$ to the plane ex 7 work



The parallelogram has area $|\vec{u}\times\vec{v}|$.

so $\text{Area} = |\langle -6, 12, -6\rangle|$

$= \sqrt{72+144}$

$= \sqrt{226}$

And $\triangle ABC$ has area $\dfrac{\sqrt{226}}{2}$