12.3 : The Dot Product

We learned to ADD vectors & mult them by scalars – how do you multiply vectors?

Defn: The Dot (or scalar) product

If $\vec{u} = \langle u_1, u_2, \ldots, u_n \rangle$ & $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$

then $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$.

ex1: Find

a) $\langle 1, -1 \rangle \cdot \langle -2, 3 \rangle = -5$

b) $(5\vec{i} - 8\vec{j} + 13\vec{k}) \cdot (2\vec{i} + 3\vec{j} - 5\vec{k}) = -79$

Q: why the "scalar" product.

Properties of the Dot product

If $\vec{u}, \vec{v}, \vec{w}$ are vectors and $c \in \mathbb{C}$, then

(1) $\vec{u} \cdot \vec{u} = \|\vec{u}\|^2$

(2) $\vec{u} \cdot (\vec{v} + \vec{w}) = u \cdot v + u \cdot w$

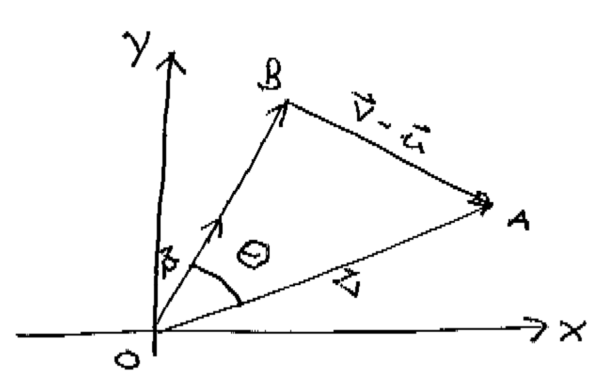
(3) $\vec{0} \cdot \vec{u} = 0$

(4) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

(5) $(c\,\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v})$

So, what is the dot product used for?

Geometrically, the dot product relates to the angle between vectors. (in $\mathbb{R}^2$)



<u>Thm</u>: If θ is the angle between $\vec{u}$ and $\vec{v}$, then $\vec{u}\cdot\vec{v} = |\vec{u}|\,|\vec{v}|\cos\theta$

<u>Proof</u> (use the law of cosines)

$$|\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}|\,|\vec{v}|\cos\theta = |\vec{v}-\vec{u}|^2$$
$$= (\vec{v}-\vec{u})\cdot(\vec{v}-\vec{u})^2$$
$$= |\vec{v}|^2 + |\vec{u}|^2 - 2(\vec{u}\cdot\vec{v})$$

$$\Rightarrow\ -2|\vec{u}|\,|\vec{v}|\cos\theta = -2\,\vec{u}\cdot\vec{v}$$
$$\Rightarrow\ \vec{u}\cdot\vec{v} = |\vec{u}|\,|\vec{v}|\cos\theta \quad \blacksquare$$

<u>ex2</u>: If vectors $\vec{u}$ & $\vec{v}$ have respective magnitudes of 7 and 5 w/ angle between of $\frac{\pi}{3}$, find $\vec{u}\cdot\vec{v}$ <u>sol</u>: $\vec{u}\cdot\vec{v} = 5\cdot 7\cos\frac{\pi}{3}$

<u>Cor</u>: If θ is the angle between vectors $\vec{u}$ & $\vec{v}$, then $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}|\,|\vec{v}|}$

<u>ex3</u>: Find the angle between $\langle 1,2,3\rangle$ & $\langle 4,5,6\rangle$

<u>sol</u>: $|\vec{u}| = \sqrt{14}$; $|\vec{v}| = \sqrt{77} \Rightarrow \arccos\left(\frac{32}{\sqrt{14}\sqrt{77}}\right)$

* <u>Cor</u>: $\vec{u}$ and $\vec{v}$ are orthogonal iff $\vec{u}\cdot\vec{v} = 0$

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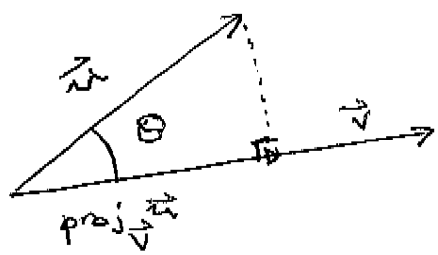
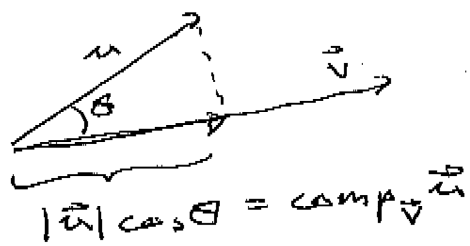
Vector projections In engineering, we frequently project vectors onto eachother.



write out $\text{proj}_{\vec{v}}\vec{u}$ and $\text{comp}_{\vec{v}}\vec{u}$

The vector $\text{proj}_{\vec{v}}\vec{u}$ has magnitude $\text{comp}_{\vec{v}}\vec{u}$

1st: Find $\text{comp}_{\vec{v}}\vec{u}$ $\qquad \vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$



$$\Rightarrow \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|} = \text{comp}_{\vec{v}}\vec{u}$$

2nd: Find $\text{proj}_{\vec{v}}\vec{u}$

A unit vector in the direction of $\vec{v}$ is $\frac{\vec{v}}{\|\vec{v}\|}$.

So, a vector in the direction of $\vec{v}$ w/ length $\text{comp}_{\vec{v}}\vec{u}$ is $\text{proj}_{\vec{v}}\vec{u} = \frac{(\text{comp}_{\vec{v}}\vec{u})\vec{v}}{|\vec{v}|} = \frac{(\vec{u}\cdot\vec{v})\vec{v}}{|\vec{v}|^2}$

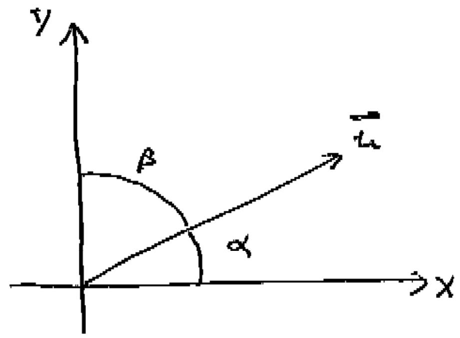
ex4: Find the projection of $\langle 1,2,3\rangle$ onto $\langle 4,5,6\rangle$

Sol: $\vec{u}\cdot\vec{v} = 32$

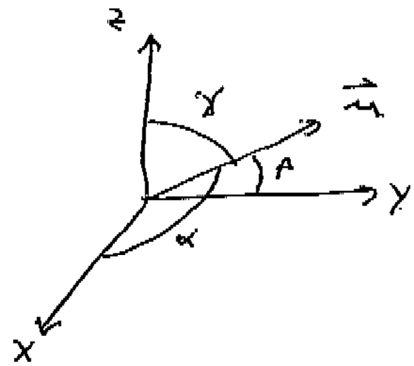
AND $|\vec{v}| = \sqrt{77}$

so $\text{proj}_{\vec{v}}\vec{u} = \frac{32}{77}\langle 4,5,6\rangle$

Direction Cosines



OR



Find α, β, and γ (or cos α, cos β, cos γ) by using cor 1 and the appropriate second vector $(\vec{i}, \vec{j}, \vec{k}) \Rightarrow \langle \cos\alpha, \cos\beta \rangle = \left\langle \frac{u_1}{|\vec{u}|}, \frac{u_2}{|\vec{u}|} \right\rangle = \frac{\vec{u}}{|\vec{u}|}$

In other words, the unit vector $\frac{\vec{u}}{|\vec{u}|}$ gives the direction cosines.

NOTE: $0 \le \alpha, \beta, \gamma \le \pi$.

Ex 5: Find the direction cosines of $\langle 1, 2, 3 \rangle$.

sol: $\langle \cos\alpha, \cos\beta, \cos\gamma \rangle = \left\langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle$