11.10: Taylor and Maclaurin Series

We will begin w/the how and then move to the why later.

Explore: $\sin(x) \approx x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots$

or $\sin(x) = 0 + x - \frac{0 \cdot x^2}{2} + \frac{-x^3}{6} + \frac{0x^4}{24} + \frac{x^5}{120} - \ldots$

where do the coefficients $\{0, 1, 0, -\frac{1}{6}, 0, \frac{1}{120}, \ldots\}$ come from?

The Maclaurin Series

If f has a power series expansion at $x=0$, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n x^n, \quad |x| < R$$

then its coefficients are given by $c_n = \frac{f^{(n)}(0)}{n!}$

and $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots$

Ex1: Find the Maclaurin Series for $f(x) = e^x$ ...

Don't forget the "If f has a..."

Ex2: Find the Maclaurin Series for $g(x) = \cos(x)$.

Ex3: Find the Maclaurin Series for $h(x) = \sin(x)$.

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Ex4: Find the Maclaurin Expansion for $f(x) = \ln(x)$.

ARGH!!!

Theorem: If f has a power series expansion at $x=a$, that is, if $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$, $|x-a|<R$ then its coefficients are given by the formula $c_n = \frac{f^{(n)}(a)}{n!}$.

So, $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$

Ex4 rev: Find the Tayor Expansion for $f(x) = \ln(x)$ around $x=1$.

$f(1) = 0$

$f'(x) = \frac{1}{x}$, $f'(1) = 1$

$f''(x) = -x^{-2}$, $f''(1) = -1$

$f^{(3)}(x) = 2x^{-3}$, $f^{(3)}(1) = 2$

$$\ln(x) = \cancel{\frac{1}{0!}(x-1) - \frac{1}{1!}(x-1)^2 + .2}$$

$$= \frac{1}{0!}(x-1) - \frac{1}{2!}(x-2)^2 + \frac{2}{3!}(x-2)^3 - \frac{6}{4!}(x-1)^4 + \dots$$

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Ex5: Evaluate $\int e^{-x^2}\,dx$ as an infinite series.

Ex6: Find the MacLaurin Series for $e^x\cos(x)$.

Ex7: Find the MacLaurin Series for $\cot(x)$.

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<u>Derivation</u> of the Maclaurin Series Formula.

Suppose $f(x)$ has a power series representation of the form $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$

we need to find the coefficients $c_0, c_1, c_2, \dots$

$$f(x) = c_0 + c_1 x + c_2 x^2 + \dots \Rightarrow f(0) = c_0$$

$$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \dots \Rightarrow f'(0) = c_1$$

$$f''(x) = 2c_2 + 6c_3 x + 12 c_4 x^2 + \dots \Rightarrow f''(0) = 2c_2$$

$$f^{(3)}(x) = 6c_3 + 24 c_4 x + 60 c_5 x^2 + \dots \Rightarrow f^{(3)}(0) = 6c_3$$

$$\vdots$$

$$f^{(n)}(x) = n!\,c_n + \underline{(n+1)!}\,c_{n+1} x + \frac{(n+2)!}{2} c_{n+2} x^2 \Rightarrow f^{(n)}(0) = n!\,c_n$$

solving for the coefficients, we have...

$$c_0 = f^{(0)}(0), \quad c_1 = f^{(1)}(0), \quad c_2 = \frac{f^{(2)}(0)}{2}, \quad c_3 = \frac{f^{(3)}(0)}{3!}, \dots, c_n = \frac{f^{(n)}(0)}{n!}$$

and thus $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ or more generally

<u>Theorem</u>: If f has a power series expansion at $x = a$, then $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$, $|x-a| < R$.

Taylor's Formula assumes the existance of a power series representation, now we must show that such an expansion exists.

* <u>Theorem</u>: If $f(x) = T_n(x) + R_n(x)$, where T_n is the n^{th} degree Taylor poly of f at $x=a$ and $\lim_{n\to\infty} R_n(x) = 0$ for $|x-a| < R$, then f is equal to the sum of its Taylor series on $|x-a| < R$.

The preceding Theorem is a pain to apply, so

<u>Taylor's Inequality</u>: If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then $|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$ for $|x-a| \leq d$.

<u>Ex1</u>: Prove that $\cos(x)$ is equal to the sum of its MacLaurin Series.

Since $f^{(n+1)}(x) = \pm\sin(x)$ or $\pm\cos(x)$, we have that $|f^{(n+1)}(x)| \leq 1 \quad \forall x \in \mathbb{R} \Rightarrow |R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$

and $\lim_{n\to\infty} |R_n(x)| \leq \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$. By the squeeze theorem, it follows that $\lim_{n\to\infty} R_n = 0$ and our claim is proved.

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* □ <u>proof of Theorem.</u>

Let $T_N(x) = \sum_{i=0}^{N} \frac{f^{(i)}(a)}{N!}(x-a)^i$ be the N^{th} degree Taylor polynomial of f at $x=a$. f is the sum of its Taylor series if $f(x) = \lim_{N\to\infty} T_N(x)$.

If $R_N(x) = f(x) - T_N(x)$ so $f(x) = T_N(x) + R_N(x)$, then $R_N(x)$ is the remainder of the Taylor series.

If $\lim_{N\to\infty} R_N(x) = 0$, then

$$\lim_{N\to\infty} T_N(x) = \lim_{N\to\infty}\left[f(x) - R_N(x)\right]$$
$$= f(x) - \lim_{N\to\infty} R_N(x)$$
$$= f(x)$$ so the theorem is proved ■

<u>Ex 2</u>: Find the Taylor series for $f(x) = \frac{1}{\sqrt{x}}$ at $x=9$.

can we show $R_N(x) = 0$?