12.8 : Power Series

A power series is a fct of the form $\sum_{n=0}^{\infty} c_n x^n$ whose domain is the set of all x's for which the series converges.

<u>NOTE:</u> c_n's are coefficients, x is the variable.

<u>Ex1:</u> $\sum_{n=0}^{\infty} c \cdot x^n = c + cx + cx^2 + \ldots$

This is a geometric series which converges when $|x| < 1$. and $\sum_{n=0}^{\infty} c x^n = \frac{c}{1-x}$, $|x| < 1$.

<u>Generally</u>: The power series centered at $x = a$ is $\sum_{n=0}^{\infty} c_n (x-a)^n$.

<u>NOTE:</u> When $x = a$, we say $(x-a)^0 = 1$.

<u>Ex 2:</u> When does $\sum_{n=0}^{\infty} n!\, x^n$ converge?

using the ratio test, we have $\lim_{n \to \infty} \frac{(n+1)!\, x^{n+1}}{n!\, x^n}$

$\hookrightarrow = \lim_{n \to \infty} (n+1)x$. This limit has magnitude less than 1 iff $x = 0$. Otherwise the limit diverges. So the serie converges iff $x = 0$.

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Ex 3: For what values of x does $\sum_{n=1}^{\infty} \frac{x^n}{n 3^n}$ converge?

* use the ratio test .... $|x| < 3$.
* check endpoints: $x = 3$ divergent
  $x = -3$ convergent.

Since the series converges on $[-3, 3)$, we say the interval of convergence is $[-3, 3)$

Review: What is the interval of convergence in Ex 1 and Ex 2?

Ex 4: Find the interval of convergence of the series $\sum_{n=0} \frac{x^n}{n!}$

Theorem: For a given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ there are only three possibilities.

i) converges only at $x = a$.
ii) converges for all x.
iii) $\exists R > 0$ s.t. the series converges when $|x - a| < R$ and diverges when $|x - a| > R$. We call R the radius of convergence.

NOTE: Radius of convergence vs. Interval of convergence

Ex 5: Find the ROC and I.O.C. of $\sum_{n=2}^{\infty} \frac{x^n}{\ln(n)}$.

Ex 6: Find the ROC and I.O.C. of $\sum_{n=0}^{\infty} \sqrt{n} \left(\frac{x-1}{2}\right)^n$

Show Cungus example of a drum membrane.

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ex5 rev: Ratio Test.

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{\ln(n+1)}\cdot\frac{\ln(n)}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x\ln(n)}{\ln(n+1)}\right|$$

$$= |x|\lim_{n\to\infty}\frac{\ln(n)}{\ln(n+1)}$$

$$\overset{H}{=} |x|\lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{1}{n+1}}$$

$$= |x|$$

so, absolutely convergent when $|x|<1$. $(R=1)$

check endpoints.

$x=1$: $\displaystyle\sum_{n=2}^{\infty}\frac{1}{\ln n} > \underbrace{\sum_{n=2}^{\infty}\frac{1}{n}}_{\text{Divergent.}}$ (obvious that $\frac{1}{\ln n} > \frac{1}{n}$ since $n > \ln n$?)

$x=-1$: $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{\ln n}$

alternating series

$\frac{1}{\ln n}$ is decreasing ($\ln(n)$ is increasing)

$\displaystyle\lim_{n\to\infty}\frac{1}{\ln(n)} = 0$

so, convergent by the alt series test.

Radius of convergence: $R=1$

Interval of convergence: IOC: $[-1, 1)$

ex6 rev: Ratio Test

$$\lim_{n\to\infty}\left|\frac{\sqrt{n+1}\left(\frac{x-1}{2}\right)^{n+1}}{\sqrt{n}\left(\frac{x-1}{2}\right)^{n}}\right| = \lim_{n\to\infty}\left|\sqrt{\frac{n+1}{n}}\left(\frac{x-1}{2}\right)\right|$$

$$= \left|\frac{x-1}{2}\right|$$

So, abs. conv when $\left|\frac{x-1}{2}\right| < 1$

$$\Rightarrow -2 < x-1 < 2$$

$$\Rightarrow -1 < x < 3$$

check endpts.

$x=3$: $\sum_1 \sqrt{n}$ Diverges.

$x=-1$: $\sum_1 \sqrt{n}\,(-1)^n$ Diverges

ROC: $R=2$

IOC: $(-1, 3)$.