11.5
1/3

11.5: Alternating Series

examples of alternating series

form

i) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$ $(-1)^{n+1} b_n,\ b_n > 0$

ii) $-1 + 1 + -1 + +1 + -1 + 1 - 1 \dots$ $(-1)^n b_n,\ b_n > 0.$

The alternating series test

If $\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - \dots \quad (b_n > 0)$

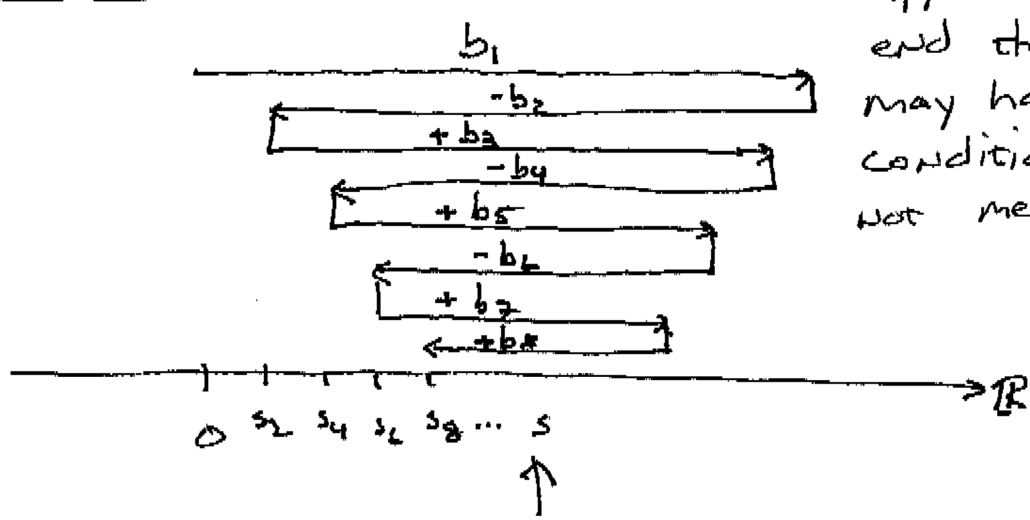
satisfies the conditions $\underbrace{b_{n+1} \le b_n \ \forall n}_{\text{magnitude decreasing (monotonic)}}$ and

$\underbrace{\lim_{n\to\infty} b_n = 0}_{\text{decreases to zero}}$, then the series converges.

Note: $b_{n+1} = b_n$ is ok

Note: 3 examples are appended on the end that show what may happen if the conditions are not met.

The picture



11.5
2/3

☐ proof.

Recall from 11.1 #72 that if $\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n+1} = L$ that $\lim_{n\to\infty} a_n = L$.

So, notice

$$s_2 = b_1 - b_2 \geq 0$$
$$s_4 = s_2 + (b_3 - b_4) \geq s_2$$
$$s_6 = s_4 + (b_5 - b_6) \geq s_4$$
$$\vdots$$
$$s_{2n} = s_{2n-2} + (\quad) \geq s_{2n-2}$$

So, $\{s_{2n}\}$ is increasing.

Now $s_{2n} = b_1 - (b_2 - b_3) - \ldots - (b_{2n-2} - b_{2n-1}) - b_{2n} < b_1$

So $\{s_{2n}\}$ is increasing and bounded above. Hence it converges. and $\lim_{n\to\infty} s_{2n} = s$.

Now
$$\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} (s_{2n} + b_{2n+1})$$
$$= \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} b_{2n+1}$$
$$= s + 0$$
$$= s$$

Since the even & odd partial sums converge to s, $\sum_{n=1}^{\infty} (-1)^{n+1} b_n = s$. ∎

11.5
3/3

Ex1: Test for convergence: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

Ex2: Test for convergence: $\sum_{n=1}^{\infty} (-1)^{n+1}$.

~~Ex3~~: What do I do if I can't easily see that $\{b_n\}$ is decreasing?

~~#~~1 What if the series begins by increasing?

Estimating Sums (Friendly)

From the picture and proof, the sum s always lives between s_n and s_{n+1}. So $|s - s_n| \leq |s_{n+1} - s_n| = b_{n+1}$. So if $s = \sum_{n=1}^{\infty} (-1)^{n+1} b_n$

is the sum of an alternating series and $0 \leq b_{n+1} \leq b_n$ and $\lim_{n\to\infty} b_n = 0$, then

$|R_n| = |s - s_n| \leq b_{n+1}$.

Ex4: Approx $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ to 4 places.

Example: A sequence $\{b_n\}$ that is decreasing but $\lim_{n\to\infty} b_n \neq 0$, and diverges

$$\sum_{n=1}^{\infty} \underbrace{\left(1+\frac{1}{n^2}\right)}_{b_n}(-1)^n$$

(1) $\{b_n\}$ is decreasing

(2) $\lim_{n\to\infty} b_n = 1$

This series diverges by the test for divergence.

Example: A sequence $\{b_n\}$ that is not decreasing, and $\lim_{n\to\infty} b_n = 0$, and diverges.

$$\sum_{n=1}^{\infty}(-1)^n b_n \quad \text{where} \quad b_n = \begin{cases} \frac{1}{n}, & n \text{ is even} \\ \frac{1}{n^2}, & n \text{ is odd} \end{cases}$$

$$\Rightarrow \sum_{n=1}^{\infty}(-1)^n b_n = -1 + \frac{1}{2} - \frac{1}{9} + \frac{1}{4} - \frac{1}{25} + \dots$$

$$= -1\left(1+\frac{1}{9}+\frac{1}{25}+\dots\right) + \left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\dots\right)$$

$$= -1\underbrace{\left(1+\frac{1}{9}+\frac{1}{25}+\dots\right)}_{\text{converges by comparison to } \sum \frac{1}{n^2}} + \frac{1}{2}\underbrace{\left(1+\frac{1}{2}+\frac{1}{3}+\dots\right)}_{\text{divergent harmonic series.}}$$

So, the series diverges (sum of convergent & divergent series)

Example: A sequence $\{b_n\}$ that is not decreasing and $\lim_{n\to\infty} b_n = 0$ and converges.

$$\sum_{n=1}^{\infty}(-1)^n b_n \quad \text{where} \quad b_n = \begin{cases} \frac{1}{n^3}, & n \text{ is even} \\ \frac{1}{n^2}, & n \text{ is odd.} \end{cases}$$

A similar argument to the previous example, but once seperated, both series converge — so convergent