$\boxed{36}$ Claim: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln(2)$.

Define

$\circ \sum_{k=1}^{n} \frac{1}{k} = h_n$

$\circ \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} = S_n$

$\square$ proof.

$\Uparrow$ We begin by using induction to prove that $S_{2n} = h_{2n} - h_n$.

when $k=1$, $S_2 = 1 - \frac{1}{2} = \frac{1}{2} = 1 + \frac{1}{2} - 1 = h_2 - h_1$.

assume $S_{2(n-1)} = h_{2(n-1)} - h_{n-1}$ this is the inductive step where we assume true for $k = n-1$.

Now, if $k = n$, $S_{2n} = S_{2(n-1)} + \frac{1}{2n-1} - \frac{1}{2n}$

$= S_{2(n-1)} + \frac{1}{2n-1} + \frac{1}{2n} - \frac{2}{2n}$

$= S_{2(n-1)} + \frac{1}{2n-1} + \frac{1}{2n} - \frac{1}{n}$

by assumption. $= h_{2(n-1)} - h_{n-1} + \frac{1}{2n-1} + \frac{1}{2n} - \frac{1}{n}$

$= \left(h_{2(n-1)} + \frac{1}{2n-1} + \frac{1}{2n}\right) - \left(h_{n-1} + \frac{1}{n}\right)$

$= h_{2n} - h_n$ ■

* proved ~~given~~ in 11.3 #38.

Moving on, we are given that $^{*}\lim_{n\to\infty}(h_n - \ln(n)) = \gamma$ and $\lim_{n\to\infty}(h_{2n} - \ln(2n)) = \gamma$. So, we have that $\lim_{n\to\infty}(h_{2n} - \ln(2n)) = \lim_{n\to\infty}(h_n - \ln(n)) = \gamma$; since the limits converge, we can combine the limits and have $\lim_{n\to\infty}\left[(h_{2n} - \ln(2n)) - (h_n - \ln(n))\right] = 0$

$\Rightarrow \lim_{n\to\infty}\left[h_{2n} - h_n - (\ln(2n) - \ln(n))\right] = 0$

$\Rightarrow \lim_{n\to\infty}\left[S_{2n} - \ln\left(\frac{2n}{n}\right)\right] = 0$

$\downarrow$

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continuing on

$$\Rightarrow \lim_{n\to\infty} \left[S_{2n} - \ln(2)\right] = 0$$

$$\Rightarrow \lim_{n\to\infty} S_{2n} - \lim_{n\to\infty} \ln(2) = 0$$

$$\Rightarrow \lim_{n\to\infty} S_{2n} - \ln(2) = 0$$

$$\Rightarrow \lim_{n\to\infty} S_{2n} = \ln(2).$$

thus the even partial sums converge.

now $\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} \left(S_{2n} + \frac{1}{2n+1}\right)$

$$= \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} \frac{1}{2n+1}$$

$$= \ln(2) + 0.$$

$$= \ln(2).$$

since $\lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} S_{2n+1} = \ln(2)$, we have that $\lim_{n\to\infty} S_n = \ln(2)$ by 11.1 #72a. ∎