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11.2: Series

In this section, we will begin our study in adding the terms in an infinite sequence $\{a_n\}_{n=1}^{\infty}$

Examples:

a) $1+2+3+4+\ldots = \sum_{n=1}^{\infty} n$

b) $1+1+1+\ldots = \sum_{n=1}^{\infty} 1$

c) $1-1-1+1+1-1-1+1+1-\ldots$

d) $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots = \sum_{n=1}^{\infty} 1\left(\frac{1}{2}\right)^{n-1}$ (a geometric ~~sequence~~ series)

e) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots = \sum_{n=1}^{\infty} \frac{1}{n}$ (the harmonic series).

each of these is an infinite series (or series)

How do we evaluate an infinite series?

Defn: Given $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots$, let

$s_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + \ldots + a_k$ be the k^{th} partial sum.

If $\{s_n\}_{n=1}^{\infty}$ is convergent, that is

$\lim_{n\to\infty} s_n = s$ (a real number), then $\sum_{n=1}^{\infty} a_n = s$.

otherwise, we say the series is divergent.

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The Geometric Series : $a + ar + ar^2 + \ldots$

If $S_k = a + ar + ar^2 + \ldots ar^{k-1}$

and $rS_k = ar + ar^2 + \ldots + ar^k$

$\Rightarrow S_k - rS_k = a - ar^k$

$\Rightarrow S_k(1-r) = a(1-r^k)$

$\Rightarrow S_k = \dfrac{a(1-r^k)}{1-r}$

Now $\lim_{k\to\infty} S_k = \lim_{k\to\infty} \dfrac{a(1-r^k)}{1-r} = \begin{cases} \dfrac{a}{1-r}, & -1<r<1 \\ \text{diverges}, & \text{else} \end{cases}$

So $\sum_{n=1}^{\infty} ar^{n-1} = \dfrac{a}{1-r}$ when $|r| < 1$.

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refer to (ex1.d).

ex2: $\sum_{n=1}^{\infty} \left(7^{-n+4}\, 2^{3n+2}\right)$

ex3: Rationalize $1.234\overline{34}$

ex4: $\sum_{n=0}^{\infty} x^n = \dfrac{1}{1-x}$ when $|x|<1$ (note: w/series, $0^0 = 1$).

(Find the domain of $\sum_{n=1}^{\infty} x^n$)

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Ex 5:
$$\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3} = \sum_{n=1}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+3}\right)$$
$$= \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\frac{1}{3} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}\right) + \dots$$
$$= \frac{5}{6} \quad \text{(Telescoping Series)}.$$

Ex 6: show that the harmonic series diverges.

$$s_1 = 1$$
$$s_2 = 1 + \frac{1}{2}$$
$$s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{2}{2}$$
$$s_8 = 1 + \frac{1}{2} + (\quad) + \left(\frac{1}{5} + \dots + \frac{1}{8}\right) > 1 + \frac{3}{2}$$
$$s_{2^n} > 1 + \frac{n}{2}$$

Thm: If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

$\square$ proof, Assume $\sum_{n=1}^{\infty} a_n = S$.

Let $S_n = a_1 + a_2 + \dots + a_n$

$\Rightarrow \quad a_n = S_n - S_{n-1}$

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1}) \quad \text{(both convergent to } S)$$
$$= \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$
$$= S - S$$
$$= 0$$

Q: Does $\lim_{n\to\infty} a_n = 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ converges?

NOTE: The contrapositive of the previous Thm,

If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_1 a_n$ diverges.

(Test for Divergence)

NOTE: w/ $\sum_1 a_n$ we have two associated sequences $\{a_n\}$ and $\{s_n\}$

Ex 7: Test $\displaystyle\sum_{n=1}^{\infty} \frac{18n^2}{(3n+1)(n-2)}$ for convergence.

Q: what does $\lim_{n\to\infty} a_n = 0$ imply about $\sum_1 a_n$ (nothing)

Thm: If $\sum_1 a_n$ & $\sum_1 b_n$ are convergent and c is a constant, then $\displaystyle\sum_{n=1}^{\infty}(ca_n \pm b_n) = c\sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$

□ proof

Let $s_n = \sum_{i=1}^{n} a_i$; $s = \sum_{n=1}^{\infty} a_n$; $t_n = \sum_{i=1}^{n} b_i$; $t = \sum_{i=1}^{\infty} b_i$

$$\sum_{n=1}^{\infty}(ca_n \pm b_n) = \lim_{n\to\infty} \sum_{i=1}^{n}(ca_i \pm b_i)$$
$$= \lim_{n\to\infty}\left(\sum_{i=1}^{n} ca_i \pm \sum_{i=1}^{n} b_i\right)$$
$$= \lim_{n\to\infty}\left(c\sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i\right)$$
$$= \lim_{n\to\infty} c\sum_{i=1}^{n} a_i \pm \lim_{n\to\infty}\sum_{i=1}^{n} b_i$$

$$= c \lim_{n\to\infty} \sum_{i=1}^{n} a_i \pm \lim_{n\to\infty} \sum_{i=1}^{n} b_i \quad \text{(they are convergent)}$$

$$= c \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$$

<u>ex 8</u> $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{n^2+4n+3} + 1 \cdot \left(\frac{1}{2}\right)^{n-1} \right)$

(see ex 1 & ex 5).