

MONOTONIC SEQUENCE THEOREM

Definition A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leq M \quad \text{for all } n \geq 1$$

It is **bounded below** if there is a number m such that

$$m \leq a_n \quad \text{for all } n \geq 1$$

If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

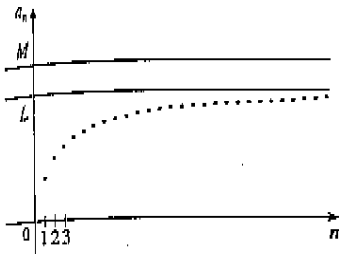


FIGURE 12

For instance, the sequence $a_n = n$ is bounded below ($a_n > 0$) but not above. The sequence $a_n = n/(n + 1)$ is bounded because $0 < a_n < 1$ for all n .

We know that not every bounded sequence is convergent [for instance, the sequence $a_n = (-1)^n$ satisfies $-1 \leq a_n \leq 1$ but is divergent from Example 6] and not every monotonic sequence is convergent ($a_n = n \rightarrow \infty$). But if a sequence is both bounded and monotonic, then it must be convergent. This fact is proved as Theorem 11, but intuitively you can understand why it is true by looking at Figure 12. If $\{a_n\}$ is increasing and $a_n \leq M$ for all n , then the terms are forced to crowd together and approach some number L .

The proof of Theorem 11 is based on the **Completeness Axiom** for the set \mathbb{R} of real numbers, which says that if S is a nonempty set of real numbers that has an upper bound M ($x \leq M$ for all x in S), then S has a **least upper bound** b . (This means that b is an upper bound for S , but if M is any other upper bound, then $b \leq M$.) The Completeness Axiom is an expression of the fact that there is no gap or hole in the real number line.

11 Monotonic Sequence Theorem Every bounded, monotonic sequence is convergent.

Proof Suppose $\{a_n\}$ is an increasing sequence. Since $\{a_n\}$ is bounded, the set $S = \{a_n \mid n \geq 1\}$ has an upper bound. By the Completeness Axiom it has a least upper bound L . Given $\epsilon > 0$, $L - \epsilon$ is not an upper bound for S (since L is the least upper bound). Therefore

$$a_N > L - \epsilon \quad \text{for some integer } N$$

But the sequence is increasing so $a_n \geq a_N$ for every $n > N$. Thus, if $n > N$ we have

$$a_n > L - \epsilon$$

so

$$0 \leq L - a_n < \epsilon$$

since $a_n \leq L$. Thus

$$|L - a_n| < \epsilon \quad \text{whenever } n > N$$

so $\lim_{n \rightarrow \infty} a_n = L$.

A similar proof (using the greatest lower bound) works if $\{a_n\}$ is decreasing.