11.1: Sequences

Def: A sequence is a list w/ defined order $a_1, a_2, a_3, \ldots, a_n, \ldots$

Notation: $\{a_1, a_2, \ldots\}$

$\{a_n\}$

$\{a_n\}_{n=1}^{\infty}$

ex1: The 3 notations

a) $\left\{\frac{n-1}{n+1}\right\}_{n=1}^{\infty}$

b) $a_n = \frac{(-1)^n 2^n}{n+2}$

c) $\left\{0, \frac{1}{2}, \frac{\sqrt{3}}{2}, 1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, -\frac{1}{2}, \ldots\right\}$

ex2: Find a formula for

$\left\{\frac{1\cancel{0}}{1\cancel{0}}, \frac{\cancel{9}2}{1}, \frac{\cancel{8}4}{2}, \frac{\cancel{16}8}{6}, \frac{\cancel{32}16}{24}, \ldots, \frac{2^n}{n!}\right\}$

ex3: Other sequences

a) $\{3, 1, 4, 1, 5, 9, \ldots\}$

b) $\{1, 1, 2, 3, 5, 8, \ldots\}$

Def. A sequence $\{a_n\}$ has the limit L and we write $\lim_{n\to\infty} a_n = L$ if $\forall\, \epsilon > 0\ \exists\, N \in \mathbb{Z}$ s.t. $n > N \Rightarrow |a_n - L| < \epsilon$.

convergent sequence.

Thm: If $\lim_{x\to\infty} f(x) = L$ & $f(n) = a_n$ for $n \in \mathbb{Z}$, then $\lim_{n\to\infty} a_n = L$.

ex4: $\lim_{n\to\infty} \frac{1}{n^r} = 0$ for $r > 0$ since $\lim_{x\to\infty} \frac{1}{x^r} = 0$.

Def. $\lim_{n\to\infty} a_n = \infty$ if $\forall\, M > 0$, $\exists\, N \in \mathbb{Z}$ s.t. $n > N \Rightarrow a_n > M$.

Limit Laws If $\{a_n\}$ & $\{b_n\}$ are convergent sequences & c is a constant.

a) $\lim_{n\to\infty} (c\, a_n \pm b_n) = c \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$

b) $\lim_{n\to\infty} (a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$

c) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ provided $\lim_{n\to\infty} b_n \neq 0$.

d) $\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p$ provided $p, a_n > 0$.

11.1
3/4

Thm: The Squeeze Thm

If $a_n \le b_n \le c_n$ for $n > n_0$, and

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$

Thm: If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$

ex 5: Find $\lim_{n\to\infty} \frac{3n}{2n-1}$

ex 6: Find $\lim_{n\to\infty} \frac{\ln n}{n^2}$ (l'Hospital's)

ex 7: Find $\lim_{n\to\infty} \sin\left(\frac{n\pi}{6}\right)$

ex 8: Find $\lim_{n\to\infty} \frac{(-1)^n \cdot 2}{n^3}$

ex 9: Does $\lim_{n\to\infty} \frac{n!}{n^n}$ converge (use the Squeeze Thm).

Q: when does $\lim_{n\to\infty} r^n$ converge?

Def. A sequence $\{a_n\}$ is increasing if $a_{n+1} > a_{n+1}$ $\forall n \ge 1$ and decreasing if $a_{n+1} < a_n$ $\forall n \ge 1$.

Def. A sequence is monotonic if it is either increasing or decreasing.

ex10: show $\frac{n+1}{n^2}$ is decreasing.

Def. A sequence $\{a_n\}$ is bounded above if $\exists M$ s.t. $a_n \leq M \ \forall n \geq 1$.

A sequence is bounded below if $\exists m$ s.t. $a_n \geq m \ \forall n \geq 1$.

If it is bounded above & below, then it is a bounded sequence.

NOTE: completeness axiom

Thm: Monotonic Sequence Theorem: Every bounded, monotonic sequence is convergent.

The proof is a nice & proof.

MONOTONIC SEQUENCE THEOREM

10 Definition A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leqslant M \qquad \text{for all } n \geqslant 1$$

It is **bounded below** if there is a number m such that

$$m \leqslant a_n \qquad \text{for all } n \geqslant 1$$

If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence.**

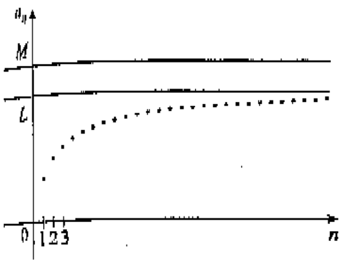

FIGURE 12

For instance, the sequence $a_n = n$ is bounded below ($a_n > 0$) but not above. The sequence $a_n = n/(n + 1)$ is bounded because $0 < a_n < 1$ for all n.

We know that not every bounded sequence is convergent [for instance, the sequence $a_n = (-1)^n$ satisfies $-1 \leqslant a_n \leqslant 1$ but is divergent from Example 6] and not every monotonic sequence is convergent ($a_n = n \to \infty$). But if a sequence is both bounded *and* monotonic, then it must be convergent. This fact is proved as Theorem 11, but intuitively you can understand why it is true by looking at Figure 12. If $\{a_n\}$ is increasing and $a_n \leqslant M$ for all n, then the terms are forced to crowd together and approach some number L.

The proof of Theorem 11 is based on the **Completeness Axiom** for the set $\mathbb{R}$ of real numbers, which says that if S is a nonempty set of real numbers that has an upper bound M ($x \leqslant M$ for all x in S), then S has a **least upper bound** b. (This means that b is an upper bound for S, but if M is any other upper bound, then $b \leqslant M$.) The Completeness Axiom is an expression of the fact that there is no gap or hole in the real number line.

11 Monotonic Sequence Theorem Every bounded, monotonic sequence is convergent.

Proof Suppose $\{a_n\}$ is an increasing sequence. Since $\{a_n\}$ is bounded, the set $S = \{a_n \mid n \geqslant 1\}$ has an upper bound. By the Completeness Axiom it has a least upper bound L. Given $\varepsilon > 0$, $L - \varepsilon$ is *not* an upper bound for S (since L is the *least* upper bound). Therefore

$$a_N > L - \varepsilon \qquad \text{for some integer } N$$

But the sequence is increasing so $a_n \geqslant a_N$ for every $n > N$. Thus, if $n > N$ we have

$$a_n > L - \varepsilon$$

so

$$0 \leqslant L - a_n < \varepsilon$$

since $a_n \leqslant L$. Thus

$$|L - a_n| < \varepsilon \qquad \text{whenever } n > N$$

so $\lim_{n\to\infty} a_n = L$.

A similar proof (using the greatest lower bound) works if $\{a_n\}$ is decreasing. ■