10.2; Calculus w/ Parametric Curves.

Overview

- Tangents
- Area
- Arclength
- SA of revolution.

Tangents and concavity

Given x(t) & y(t), to find the slope of a tangent we need $\frac{dy}{dx}$.

But $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$ when $\frac{dx}{dt} \neq 0$

To find concavity, we need $\frac{d^2y}{dx^2}$.

But $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

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Another derivation for when $x(t)$ is one-to-one (why

so, $\exists f$ s.t. $y(t) = f(x(t))$

$$\Rightarrow y'(t) = f'(x(t)) \cdot x'(t) \quad \text{chain rule}$$

$$\Rightarrow f'(x(t)) = \frac{y'(t)}{x'(t)} \longleftarrow \text{1st derivative,}$$

$$\Rightarrow f''(x(t)) \cdot x'(t) = \frac{y''(t)x'(t) - x''(t)y'(t)}{(x'(t))^2}$$

$$\Rightarrow f''(x(t)) = \frac{y''(t)x'(t) - x''(t)y'(t)}{(x'(t))^3} \longleftarrow$$

second derivative

compare this to

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$$= \frac{\frac{d}{dt}\left(\frac{dy/dt}{dx/dt}\right)}{\frac{dx}{dt}}$$

$$= \frac{\frac{d^2y}{dt}\frac{dx}{dt} - \frac{d^2x}{dt}\frac{dy}{dt}}{\left(\frac{dx}{dt}\right)^3}$$

(where did the cube come from?)

so, you can see the formulas are equivalent.

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Ex1: Sketch $x = 16 - t^2$ and $y = t^3 - 12t$

① Find the intercepts.

$x = 0$: $t = \pm 4$

$y(4) = 16 \Rightarrow (0, 16)$ @ $t = 4$

$y(-4) = -16 \Rightarrow (0, -16)$ @ $t = -4$

$y = 0$: $t = 0$ and $t = \pm\sqrt{12}$

$x(0) = 16 \Rightarrow (16, 0)$ @ $t = 0$

$$\left.\begin{array}{l} x(\sqrt{12}) = 4 \Rightarrow (4, 0) \\ x(-\sqrt{12}) = 4 \Rightarrow (4, 0) \end{array}\right\} @\ t = \pm\sqrt{12}$$

② Find critical values.

$$\frac{dy}{dx} = \frac{3t^2 - 12}{-2t}$$

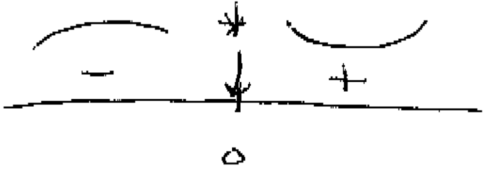
Hor. tangent @ $t = \pm 2$ $\to (12, -16)$, $\to (12, 16)$

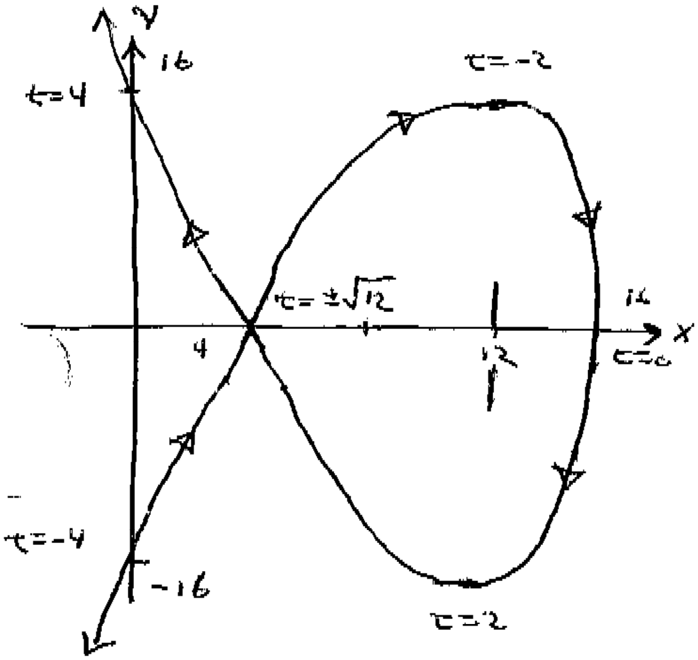
Ver. tangent @ $t = 0$

Tangent when $t = \sqrt{12}$ has neg. slope and visaversa.

③ concavity

$$\frac{d^2y}{dx^2} = \frac{3t^2 + 12}{t}$$





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Area

If a graph $(x(t), y(t))$ is never below the x-axis & the curve is traversed once as t increases from α to β, then

$$\int_a^b y\,dx = \int_\alpha^\beta y(t)\,x'(t)\,dt$$

(assuming $t=\alpha$ gives the left endpoint)

ex: $x(t) = r\cos t$ & $y(t) = r\sin t$

$$\int_0^\pi r\sin t \cdot (-r\sin t)\,dt = -r^2\int_0^\pi \sin^2 t\,dt$$

$$= -r^2\int_0^\pi \frac{1-\cos 2t}{2}\,dt$$

$$= \frac{-r^2}{2}\left(t - \frac{1}{2}\sin 2t\right)\Big|_0^\pi$$

$$= \frac{-r^2}{2}\pi$$

$$= \frac{-\pi r^2}{2}$$

Neg because $t=0$ is the right endpoint.

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Arc Length

recall, the length L of a curve C given by $y = f(x)$ on $a \leq x \leq b$ when f' is cont is given by

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

If C is also described by $(x(t), y(t))$ on $\alpha \leq t \leq \beta$ where $x'(t) > 0$ (why?), then

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_\alpha^\beta \sqrt{1+\left(\frac{dy/dt}{dx/dt}\right)^2}\, \frac{dx}{dt}\, dt$$

$$= \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\, dt \qquad (x'(t) > 0)$$

ex: Find the circ. of a circle.

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Arc Length

We already know how to find the length L of a curve C given in the form $y = F(x)$, $a \leqslant x \leqslant b$. Formula 8.1.3 says that if F' is continuous, then

$$\boxed{3} \qquad L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Suppose that C can also be described by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leqslant t \leqslant \beta$, where $dx/dt = f'(t) > 0$. This means that C is traversed once, from left to right, as t increases from α to β and $f(\alpha) = a$, $f(\beta) = b$. Putting Formula 2 into Formula 3 and using the Substitution Rule, we obtain

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\, \frac{dx}{dt}\, dt$$

Since $dx/dt > 0$, we have

$$\boxed{4} \qquad L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Even if C can't be expressed in the form $y = F(x)$, Formula 4 is still valid but we obtain it by polygonal approximations. We divide the parameter interval $[\alpha, \beta]$ into n subintervals of equal width Δt. If $t_0, t_1, t_2, \ldots, t_n$ are the endpoints of these subintervals, then $x_i = f(t_i)$ and $y_i = g(t_i)$ are the coordinates of points $P_i(x_i, y_i)$ that lie on C and the polygon with vertices $P_0, P_1, \ldots, P_n$ approximates C (see Figure 4).

As in Section 8.1, we define the length L of C to be the limit of the lengths of these approximating polygons as $n \to \infty$:

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

The Mean Value Theorem, when applied to f on the interval $[t_{i-1}, t_i]$, gives a number t_i^* in (t_{i-1}, t_i) such that

$$f(t_i) - f(t_{i-1}) = f'(t_i^*)(t_i - t_{i-1})$$

If we let $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_i - y_{i-1}$, this equation becomes

$$\Delta x_i = f'(t_i^*)\, \Delta t$$

MVT (4.2, p291)
If f is cont on $[a,b]$
and f is diff on (a,b)
Then $\exists\, c \in (a,b)$ s.t.
$f(b) - f(a) = f'(c)(b-a)$

Similarly, when applied to g, the Mean Value Theorem gives a number t_i^{**} in (t_{i-1}, t_i) such that

$$\Delta y_i = g'(t_i^{**})\, \Delta t$$

Therefore

$$\begin{aligned} |P_{i-1}P_i| &= \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} \\ &= \sqrt{[f'(t_i^*)\Delta t]^2 + [g'(t_i^{**})\Delta t]^2} \\ &= \sqrt{[f'(t_i^*)]^2 + [g'(t_i^{**})]^2}\, \Delta t \end{aligned}$$

and so

$$\boxed{5} \qquad L = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{[f'(t_i^*)]^2 + [g'(t_i^{**})]^2}\, \Delta t$$

The sum in (5) resembles a Riemann sum for the function $\sqrt{[f'(t)]^2 + [g'(t)]^2}$ but it is not exactly a Riemann sum because $t_i^* \neq t_i^{**}$ in general. Nevertheless, if f' and g' are continuous, it can be shown that the limit in (5) is the same as if t_i^* and t_i^{**} were equal, namely, ← caution

$$L = \int_\alpha^\beta \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

Thus, using Leibniz notation, we have the following result, which has the same form as (4).

Surface Area

similarly, if $(x(t), y(t))$, $\alpha \leq t \leq \beta$ is rotated about the x-axis, where x' & y' are cont & $y(t) \geq 0$, then the SA is

$$S = \int_{\alpha}^{\beta} 2\pi y \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$