9.) Find the Taylor series for $f(x) = x^{-2}$ centered at $a = 1$ and its interval of convergence. [Assume that f has a power expansion. Do not show that $R_n(x) \to 0$.]

evaluate @ $a=1$

$f^{(0)}(x) = x^{-2}$ — 1

$f^{(1)}(x) = -2x^{-3}$ — -2

$f^{(2)}(x) = 6x^{-4}$ — 6

$f^{(3)}(x) = -24x^{-5}$ — -24

$\vdots$

$f^{(n)}(x) = (-1)^n (n+1)!\, x^{-(n+2)}$ — $(-1)^n (n+1)!$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)!}{n!}(x-1)^n = \sum_{n=0}^{\infty} (-1)^n (n+1)(x-1)^n$$

Start w/ IOC: RATIO TEST

$$\lim_{n\to\infty} \left| \frac{(n+2)(x-1)^{n+1}}{(n+1)(x-1)^n} \right| = |x-1|$$

AND $|x-1| < 1 \Rightarrow 0 < x < 2$

Test end points.

$x=0$: $\sum (n+1)$ diverges

$x=2$: $\sum (-1)^n (n+1)$ diverges

IOC $= (0,2)$

ROC $= 1$

10.) Approximate $f(x) = x\ln(x)$ by a Taylor polynomial with degree 3 at the number $a = 1$. Then use Taylor's Inequality to estimate the accuracy of the approximation $f(x) \approx T_3(x)$ when $0.5 \le x \le 1.5$. Check this result by graphing $|R_3(x)|$ on the given interval and finding its maximum.

$x \ln x = f^{(0)}(x) \Big|_{x=1}$ 0

$\ln x + 1 = f^{(1)}(x) \Big|_{x=1}$ 1

$\frac{1}{x} = f^{(2)}(x) \Big|_{x=1}$ 1

$-\frac{1}{x^2} = f^{(3)}(x) \Big|_{x=1}$ -1

$+\frac{2}{x^3} = f^{(4)}(x) \Big|_{x=1}$ 2

$$T_3(x) = (x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3$$

To estimate the error, I will use Taylor's Inequality

$$\left| \frac{2}{x^3} \right| \le \frac{2}{(\frac{1}{2})^3} = 16 \text{ on } \frac{1}{2} \le x \le \frac{3}{2}$$

so $M = 16$.

$$\Rightarrow |R_3(x)| \le \frac{16}{4!}|x-1|^4 \le \underbrace{\frac{16}{24}\left(\frac{1}{2}\right)^4}_{\approx 0.0416}$$

Actual max error ≈ 0.0076

11.) Evaluate the indefinite integral $\int \tan^{-1}\left(x^2\right)dx$ as a power series. What is its radius of convergence?

$\tan^{-1}(x^2)$

$\frac{d}{dx}$

$\frac{2x}{1+x^4}$

ignore $2x$

$\frac{1}{1-(-x^4)} \longrightarrow \sum (-x^4)^n = \sum (-1)^n x^{4n}$

And recall $2x$

$\int \quad 2x \sum \frac{(-1)^n x^{4n+1}}{4n+1}$

on $|x|<1$ or $R=1$

so

$\int \tan^{-1}(x^2)\,dx$

$= 2\int \sum \frac{(-1)^n x^{4n+1}}{4n+1}\,dx$

$= 2\sum_{n=0} \frac{(-1)^n x^{4n+2}}{(4n+1)(4n+2)}$

So $R = 1$

SEE ATTACHED

12.) Find a power series representation for $f(x) = x\cdot\cos\left(x^4\right)$

$x \cos(x^4)$

$= x\left(1 - \frac{(x^4)^2}{2!} + \frac{(x^4)^4}{4!} - \frac{(x^4)^6}{6!} + \dots\right)$

$= \frac{x}{0!} - \frac{x^9}{2!} + \frac{x^{17}}{4!} - \frac{x^{25}}{6!} + \dots$

$= \sum_{n=0}^{\infty} \frac{x^{8n+1}}{n!}(-1)^n$

$$\begin{tikzcd}
\tan^{-1}(x^2) \arrow[d, "\frac{d}{dx}"'] & \sum \frac{2(-1)^n x^{4n+2}}{4n+2} = \sum \frac{(-1)^n x^{4n+2}}{2n+1} \\
\frac{2x}{1+x^4} \arrow[d, "\text{ignore } 2x"'] & \sum 2(-1)^n x^{4n+1} \arrow[u, "\int"'] \\
\frac{1}{1-(-x^4)} \arrow[r] & \sum (-x^4)^n = \sum (-1)^n x^{4n} \arrow[u, "\text{recall } 2x"']
\end{tikzcd}$$

$$\text{so} \int \tan^{-1}(x^2)\,dx = \int \sum \frac{(-1)^n x^{4n+2}}{2n+1}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n+1)(4n+3)}$$

$$
\begin{tikzcd}
\dfrac{x^2}{(1-2x)^2} \arrow[d, "\text{ignore } x^2"'] & \displaystyle\sum_{\substack{n=0 \\ \text{or } n=1}} n\, 2^{n-1} x^{n+1} \\
\dfrac{1}{(1-2x)^2} \arrow[d, "\int"'] & \dfrac{1}{2} \displaystyle\sum 2n (2x)^{n-1} \arrow[u, "\text{recall } x^2"'] \\
\dfrac{1}{2(1-2x)} \arrow[r] & \dfrac{1}{2} \displaystyle\sum (2x)^n \arrow[u, "\frac{d}{dx}"']
\end{tikzcd}
$$