

9.) Find the Taylor series for $f(x) = x^{-2}$ centered at $a = 1$ and its interval of convergence. [Assume that f has a power expansion. Do not show that $R_n(x) \rightarrow 0$.]

$f^{(0)}(x) = x^{-2}$	1	equivalent $a=1$ some w IOC: RATIO TEST $\lim_{n \rightarrow \infty} \left \frac{(n+2)(x-1)^{n+1}}{(n+1)(x-1)^n} \right = x-1 $ AND $ x-1 < 1 \Rightarrow 0 < x < 2$ Test end points. $x=0: \sum (n+1)$ diverges $x=2: \sum (-1)^n (n+1)$ diverges IOC = $(0, 2)$ ROC = 1
$f^{(1)}(x) = -2x^{-3}$	-2	
$f^{(2)}(x) = 6x^{-4}$	6	
$f^{(3)}(x) = -24x^{-5}$	-24	
\vdots		
$f^{(n)}(x) = (-1)^n (n+1)! x^{-(n+2)}$	$(-1)^n (n+1)!$	
$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)!}{n!} (x-1)^n = \sum_{n=0}^{\infty} (-1)^n (n+1) (x-1)^n$		

10.) Approximate $f(x) = x \ln(x)$ by a Taylor polynomial with degree 3 at the number $a = 1$. Then use Taylor's Inequality to estimate the accuracy of the approximation $f(x) \approx T_3(x)$ when $0.5 \leq x \leq 1.5$. Check this result by graphing $|R_3(x)|$ on the given interval and finding its maximum.

$x \ln x = f^{(0)}(x) \Big _{x=1} = 0$	$T_3(x) = (x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3$
$\ln x + 1 = f^{(1)}(x) \Big _{x=1} = 1$	To estimate the error, I will use Taylor's Inequality
$\frac{1}{x} = f^{(2)}(x) \Big _{x=1} = 1$	$\left \frac{2}{x^3} \right \leq \frac{2}{(\frac{1}{2})^3} = 16$ on $\frac{1}{2} \leq x \leq \frac{3}{2}$
$-\frac{1}{x^2} = f^{(3)}(x) \Big _{x=1} = -1$	so $M = 16$.
$+\frac{2}{x^3} = f^{(4)}(x) \Big _{x=1} = 2$	$\Rightarrow R_3(x) \leq \frac{16}{4!} x-1 ^4 \leq \frac{16}{24} \left(\frac{1}{2}\right)^4 \approx 0.041\bar{6}$

Actual max error ≈ 0.0076

11.) Evaluate the indefinite integral $\int \tan^{-1}(x^2) dx$ as a power series. What is its radius of convergence?

~~Handwritten work for problem 11, including:~~

- $\frac{d}{dx} \tan^{-1}(x^2) = \frac{2x}{1+x^4}$
- $\frac{1}{1+x^4} = \frac{1}{1-(x^4)^2} = \sum_{n=0}^{\infty} (x^4)^{2n} = \sum_{n=0}^{\infty} x^{8n}$
- $\int \frac{2x}{1+x^4} dx = \int 2x \sum_{n=0}^{\infty} x^{8n} dx = 2 \sum_{n=0}^{\infty} \int x^{8n+1} dx = 2 \sum_{n=0}^{\infty} \frac{x^{8n+2}}{8n+2}$
- Radius of convergence $R=1$.

SEE ATTACHED.

12.) Find a power series representation for $f(x) = x \cdot \cos(x^4)$

$$\begin{aligned}
 & x \cos(x^4) \\
 &= x \left(1 - \frac{(x^4)^2}{2!} + \frac{(x^4)^4}{4!} - \frac{(x^4)^6}{6!} + \dots \right) \\
 &= \frac{x}{0!} - \frac{x^9}{2!} + \frac{x^{17}}{4!} - \frac{x^{25}}{6!} + \dots \\
 &= \sum_{n=0}^{\infty} \frac{x^{8n+1}}{n!} (-1)^n
 \end{aligned}$$

$$\tan^{-1}(x^2)$$

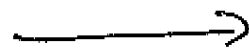
$$\frac{d}{dx} \downarrow$$

$$\frac{2x}{1+x^4}$$

ignore
2x

$$\downarrow$$

$$\frac{1}{1-(-x^4)}$$



$$\sum_1 (-x^4)^n = \sum_1 (-1)^n x^{4n}$$

$$\sum_1 \frac{2(-1)^n x^{4n+2}}{4n+2} = \sum_1 \frac{(-1)^n x^{4n+2}}{2n+1}$$

$$\uparrow \int$$

$$\sum_1 2(-1)^n x^{4n+1}$$

$$\uparrow \text{recall}$$

2x

$$\begin{aligned} \text{so } \int \tan^{-1}(x^2) dx &= \int \sum_1 \frac{(-1)^n x^{4n+2}}{2n+1} dx \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n+1)(4n+3)} \end{aligned}$$

$$\frac{x^2}{(1-2x)^2}$$

ignore
 x^2

$$\frac{1}{(1-2x)^2}$$

\int

$$\frac{1}{2(1-2x)}$$

$$\sum_{n=0}^{\infty} n 2^{n-1} x^{n+1}$$

OR $n=1$

recall x^2

$$\frac{1}{2} \sum_{n=1}^{\infty} n 2^n (2x)^{n-1}$$

$\frac{d}{dx}$

$$\frac{1}{2} \sum_{n=1}^{\infty} (2x)^n$$