SOLUTIONS TO PROBLEMS (1) THRU (9)

13.3: Arc Length and Curvature

- Arc length (and reparametrization).
- The arc length function.
- Curvature. (four definitions).
- The normal vector
- The binormal vector
- The osculating plane.
- The osculating circle (can you find its equation in two dimensions?).

13.4: Motion in Space: Velocity and Acceleration

- The velocity function.
- The acceleration function.
- Tangential and normal components of acceleration.
- You will not be tested on Kepler's Laws.

Practice Problems

1.) Determine whether $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ converges or diverges. If it converges, what is the limit?

Suppose $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = A \Rightarrow \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = A \Rightarrow \ln\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right) = \ln A$

$\Rightarrow \lim_{x\to\infty}\left(\ln\left(1+\frac{1}{x}\right)^x\right) = \ln A \Rightarrow \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = \ln A$

$\overset{H}{\Rightarrow} \lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\cdot\cancel{\frac{-1}{x^2}}}{\cancel{\frac{-1}{x^2}}} = \ln A \Rightarrow \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}} = \ln A \Rightarrow 1 = \ln A$

$\Rightarrow A = e.$

2.) Determine whether $\lim_{n\to\infty}\frac{(-7)^n}{n!}$ converges or diverges. If it converges, what is the limit?

$= \lim_{n\to\infty}\underbrace{\frac{-7}{1}\cdot\frac{-7}{2}\cdot\frac{-7}{3}\cdots\frac{-7}{7}}\cdot\underbrace{\frac{-7}{8}\cdot\frac{-7}{9}\cdots\frac{-7}{n}}_{\text{each less than }(-1)}$

$= \lim_{n\to\infty} c\left(\frac{-7}{8}\right)\cdots\left(\frac{-7}{n}\right)$ and $\lim_{n\to\infty}\frac{-7}{n} = 0$

$\Rightarrow = 0$

3.) Determine whether $\sum_{n=0}^{\infty}\frac{3^n+2^n}{6^n}$ converges or diverges. If it converges, find the sum.

$= \sum_{n=0}^{\infty}\left[\left(\frac{3}{6}\right)^n+\left(\frac{2}{6}\right)^n\right] = \sum_{n=1}^{\infty}\left(\left(\frac{1}{2}\right)^{n-1}+\left(\frac{1}{3}\right)^{n-1}\right) = \frac{1}{1-\frac{1}{2}} + \frac{1}{1-\frac{1}{3}}$

$= 2 + \frac{3}{2}$

$= 7/2.$

4.) Determine whether $\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+5}\right)$ converges or diverges. If it converges, find the sum.

$$\lim_{n\to\infty} \ln\left(\frac{n}{2n+5}\right) = \ln\left(\lim_{n\to\infty} \frac{n}{2n+5}\right) \quad \text{(since ln is cont on } n>0\text{)}$$
$$= \ln\left(\frac{1}{2}\right)$$
$$\neq 0$$

Hence, the series diverges.

5.) Determine whether $\sum_{n=1}^{\infty} \left(\frac{2}{9^n} + \frac{7}{n}\right)$ converges or diverges. If it converges, find the sum.

$$= \underbrace{\sum_{n=1}^{\infty} \frac{2}{9^n}}_{\substack{\text{geometric} \\ \text{(converges)}}} + \underbrace{\sum_{n=1}^{\infty} \frac{7}{n}}_{\substack{\text{harmonic} \\ \text{(diverges)}}}$$

so, it diverges.

6.) Find the vector equation for a line segment that joins $P(7,2,-3)$ and $Q(-3,1,4)$.

$$\vec{v} = \langle 10, 1, -7 \rangle \quad (\vec{P} - \vec{Q})$$
$$\vec{r}(t) = \langle -3, 1, 4 \rangle (1-t) + \langle 7, 2, -3 \rangle t \quad (0 \le t \le 1)$$
$$\text{OR } \vec{r}(t) = \langle -3, 1, 4 \rangle + t\langle 10, 1, -7 \rangle \quad (0 \le t \le 1)$$

7.) At what point do the curves $\vec{r}_1(t) = \langle t, 1-t, 3+t^2 \rangle$ and $\vec{r}_2(s) = \langle 3-s, s-2, s^2 \rangle$ intersect (begin by finding t and s)? Find their angle of intersection correct to the nearest degree.

$\boxed{\text{pt of intersection}}$

$t = 3 - s$

$1 - t = s - 2$

$\Rightarrow 1 - (3 - s) = s - 2$

$\Rightarrow -2 + s = s - 2$ (duh)

OR $3 + (3-s)^2 = s^2$

$\Rightarrow 3 + 9 - 6s + s^2 = s^2$

$\Rightarrow 12 = 6s$

$\Rightarrow s = 2$

AND $t = 1$

$\vec{r}_1{}'(t) = \langle 1, -1, 2t \rangle \Rightarrow \vec{r}_1{}'(1) = \langle 1, -1, 2 \rangle = \vec{u}$

$\vec{r}_2{}'(s) = \langle -1, 1, 2s \rangle \Rightarrow \vec{r}_2{}'(2) = \langle -1, 1, 4 \rangle = \vec{v}$

recall $u \cdot v = |u||v| \cos\theta$

$\vec{u} \cdot \vec{v} = -1 - 1 + 8 = 6$

$|\vec{u}| = \sqrt{1+1+4} = \sqrt{6}$ & $|\vec{v}| = \sqrt{1+1+16} = \sqrt{18}$

$\Rightarrow \theta = \arccos\left(\frac{6}{\sqrt{108}}\right)$

7.) At what point do the curves $\bar{r}_1(t) = \langle t, 1-t, 3+t^2 \rangle$ and $\bar{r}_2(s) = \langle 3-s, s-2, s^2 \rangle$ intersect (begin by finding t and s)? Find their angle of intersection correct to the nearest degree.

see previous page.

8.) Find the equations of the normal plane and osculating plane as well as the curvature of the curve $x = t,\ y = t^2,\ z = \frac{2}{3}t^3$ at the point $\left(1, 1, \frac{2}{3}\right)$.

a.) What time is it? $t = 1$

b.) What is a vector tangent to the curve (and normal to the normal plane) at the time found in part (a.)?

$$\vec{r}\,'(t) = \langle 1, 2t, 2t^2 \rangle \Rightarrow \vec{r}\,'(1) = \langle 1, 2, 2 \rangle$$

Normal Plane: $(x-1) + 2(y-1) + 2\left(z - \frac{2}{3}\right) = 0.$

c.) To find the osculating plane, you must find the tangent, normal, and binormal vectors at the time found in part (a.). This will require good algebra as well as the quotient rule.

From (b) $|\vec{r}\,'(t)| = \sqrt{1 + 4t^2 + 4t^4}$

$$= (1 + 2t^2)$$

$$\Rightarrow \vec{T}(t) = \frac{1}{1+2t^2}\langle 1, 2t, 2t^2 \rangle$$

$$\vec{T}\,'(t) = \frac{(1+2t^2)\langle 0, 2, 4t \rangle - 4t\langle 1, 2t, 2t^2 \rangle}{(1+2t^2)^2}$$

$$= \frac{1}{(1+2t^2)^2}\langle -4t, 2-4t^2, 4t \rangle$$

AND $|\vec{T}\,'(1)| = \left| \frac{1}{9}\langle -4, -2, 4 \rangle \right|$

$$= \frac{1}{9}\sqrt{16+4+16}$$

$$= \frac{6}{9} = 2/3 = |\vec{T}\,'(1)|$$

so:

$$\vec{T}(1) = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle$$

$$\vec{N}(1) = \left\langle -\frac{2}{3}, \frac{-1}{3}, \frac{2}{3} \right\rangle$$

$$\vec{T}(1) \times \vec{N}(1) = \vec{B}(1)$$

$$\vec{B}(1) = \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle$$

AND THE osculating plane

$$\frac{2}{3}(x-1) + \frac{-2}{3}(y-1) + \frac{1}{3}\left(z - \frac{2}{3}\right) = 0$$

9.) Find the tangential and normal components of the acceleration vector $\vec{r}(t) = \langle t, t^2, 3t \rangle$.

$\vec{r}'(t) = \langle 1, 2t, 3 \rangle$

$\vec{r}''(t) = \langle 0, 2, 0 \rangle$

AND $|r'(t)| = \sqrt{1+4t^2+9}$

$= \sqrt{10+4t^2}$

$a_T = \dfrac{0+4t+0}{\sqrt{10+4t^2}} = \dfrac{4t}{\sqrt{10+4t^2}}$

$a_N = \dfrac{|\langle -6, 0, 2 \rangle|}{\sqrt{10+4t^2}} = \sqrt{\dfrac{40}{10+4t^2}}$

Other cool problems for the ambitious and those with trouble sleeping YOU ARE ON YOUR OWN.

10.) Find a vector function that represents the curve of the intersection of surfaces $z = 4x^2 + y^2$ (a paraboloid) and $y = x^2$ (a parabolic cylinder).

11.) Show that if r is a vector function such that r'' exists, then: $\dfrac{d}{dt}[r(t) \times r'(t)] = r(t) \times r''(t)$.