If $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$, find $\sinh^{-1}(x)$.

$$y = \sinh(x) = \frac{e^x - e^{-x}}{2}$$

swap x & y

$$x = \frac{e^y - e^{-y}}{2}$$

solve for y.

$$\Rightarrow 2x = e^y - \frac{1}{e^y}$$

$$\Rightarrow 2xe^y = (e^y)^2 - 1$$

$$\Rightarrow 0 = (e^y)^2 - 2x(e^y)^1 - 1$$

Quadratic in Disguise.

$$\Rightarrow e^y = \frac{2x \pm \sqrt{4x^2 - 4(1)(-1)}}{2}$$

$$\Rightarrow e^y = \frac{2x \pm \sqrt{4(x^2+1)}}{2}$$

$$\Rightarrow e^y = x \pm \sqrt{x^2+1}$$

$$\Rightarrow y = \ln(x \pm \sqrt{x^2+1})$$

which do I take?

If $x - \sqrt{x^2+1}$.

claim: $x - \sqrt{x^2+1} < 0$ for all $x \in \mathbb{R}$

$\square$ proof.

$$x^2+1 > x^2$$

$$\Rightarrow \sqrt{x^2+1} > \sqrt{x^2} = |x|$$

$$\Rightarrow 0 > |x| - \sqrt{x^2+1}$$

AND

$$|x| - \sqrt{x^2+1} > x - \sqrt{x^2+1}.$$

Hence $0 > x - \sqrt{x^2+1}$

AND $\ln(x - \sqrt{x^2+1})$ has no domain $\blacksquare$

THE KEY

If $x + \sqrt{x^2+1}$.

claim: $x + \sqrt{x^2+1} > 0$ for all $x \in \mathbb{R}$ and hence $\ln(x + \sqrt{x^2+1})$ is defined for all $x \in \mathbb{R}$.

$\square$ proof.

$$x + \sqrt{x^2+1} \overset{?}{>} 0$$

$$\Rightarrow (x + \sqrt{x^2+1})\left(\frac{x - \sqrt{x^2+1}}{x - \sqrt{x^2+1}}\right) \overset{?}{>} 0$$

$$\Rightarrow \frac{x^2 - (x^2+1)}{x - \sqrt{x^2+1}} \overset{?}{>} 0$$

$$\Rightarrow \frac{-1}{x - \sqrt{x^2+1}} \overset{?}{>} 0$$

$$\Rightarrow x - \sqrt{x^2+1} < 0$$ $\blacksquare$ proved earlier.

Hence: $\boxed{\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})}$