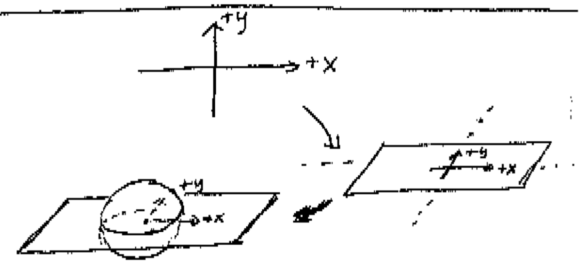
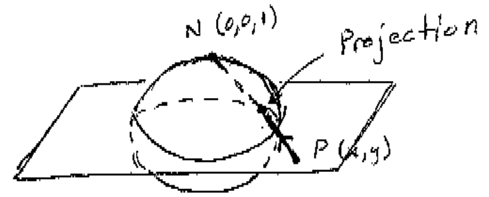
# Stereographic Projection of $\mathbb{R}^2$ onto $S^2$ (unit sphere in $\mathbb{R}^3$)




View $\mathbb{R}^2$ as $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$

$$(x, y) \longmapsto (x, y, 0).$$

Visualize the unit sphere w/ plane through it.




Let $(0,0,1)$ be $N$, the north pole.

To map any point $P=(x,y)$ from $\mathbb{R}^2$ onto $S^2$, draw the line between $N$ & $P$. Where it intersects $S^2$ is the projection of $P$.

[For example, the origin $(0,0)$ maps downward to $(0,0,-1)$.]

---

To find point on sphere, parameterize line.

$$M(t) = (x,y,0) + t(-x,-y,1) \qquad M(0) = (x,y,0) \leftarrow \text{"}P\text{"}$$

$$= ((1-t)x, (1-t)y, t) \qquad M(1) = (0,0,1) = N$$

Need $t_0$ such that point $M(t_0)$ is on sphere, i.e.

$$[(1-t_0)x]^2 + [(1-t_0)y]^2 + t_0^2 = 1$$

$$\downarrow$$

$$(x^2+y^2)(1-t_0)^2 = 1 - t_0^2$$

$t_0 \neq 1$ (not @ $N$) so $(x^2+y^2)(1-t_0) = 1 + t_0$

$$\vdots$$

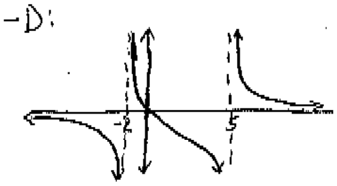
$$t_0 = \frac{x^2+y^2-1}{x^2+y^2+1} \quad \left[= \frac{\|P\|-1}{\|P\|+1}\right]$$

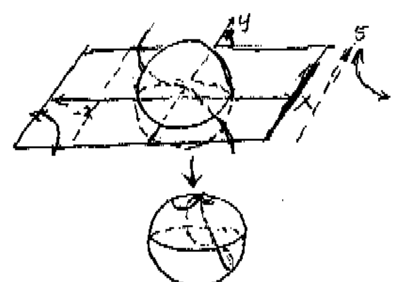
So $M(t_0) = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right)$

Thus our mapping is $(x,y) \longmapsto M(t_0)$

$R(x) = \dfrac{4x-3}{(x+2)(x-5)}$

2-D:

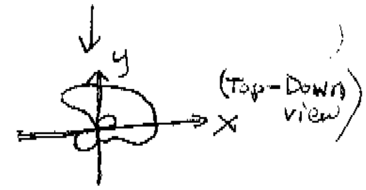



---

Can now map 2-D graphs to sphere:

$$(x, f(x)) = (x,y) \longmapsto M(t_0)$$




Or return to a 2-D figure by projecting vertically to disk (flatten sphere)

$$(x, f(x)) = (x,y) \longmapsto M(t_0) \xmapsto{\pi_{xy}} \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}\right)$$