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We concluded yesterday by stating that:

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x \quad \text{where } \Delta x = \frac{b-a}{n}$$

or more generally

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \quad \text{where } \Delta x \text{ is as above \& } x_{i-1} \le x_i^* \le x_i.$$

This allows us to state the following definition.

<u>Defn</u>: The Definite Integral.

If f is a continuous fct on $a \le x \le b$, $\Delta x = \frac{b-a}{n}$, ~~and~~ $x_i = a + i\Delta x$ for $i \in \{0, 1, 2, \dots, n\}$, and $x_{i-1} \le x_i^* \le x_i$. Then the definite integral of f from a to b is:

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)dx.$$

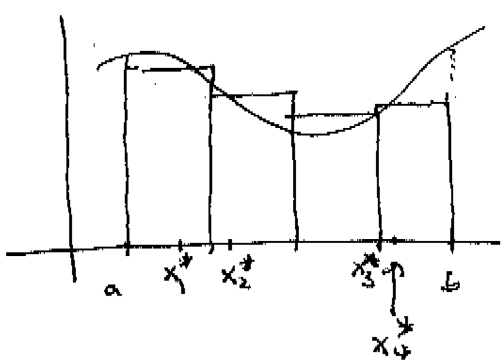
$\boxed{\text{Notation}}$

$\int$	integral sign	$f(x)$	integrand
a, b	are the lower and upper limits of integration.	dx	dx

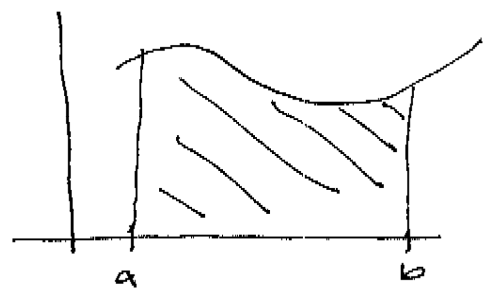
Note: while the derivative is a function, the definite integral is a number and does not depend on x.

Note: We call $\sum_{i=1}^{n} f(x_i^*)\Delta x$ the Riemann sum & our defn of the integral is for the Riemann Integral

So ...



$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$



$$\int_a^b f(x)\,dx$$

Ex1: Express $\lim_{n\to\infty} \sum_{i=1}^{n} (x_i - x_i^2 \ln(x_i))\Delta x$ as an integral on $[1, e]$

Ex2: Express $\int_2^4 x^3\,dx$ as the limit of its Riemann sum.

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Good stuff to know

a) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

b) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

c) $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

d) $\sum_{i=1}^{n} c = nc$

e) $\sum_{i=1}^{n} c \cdot a_i = c \sum_{i=1}^{n} a_i$

f) $\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

Ex 3: Evaluate the Riemann sum for $f(x) = x^2 - 3x$ taking sample points to be left end points and $a = 1$, $b = 5$, and $n = 4$.

Ex 4: Evaluate $\int_0^3 (x^2 - 3x)\,dx$

Ex 5: Set up an expression for $\int_0^{\pi} \sin(x)\,dx$ as a limit of sums.

Evaluate the following geometrically.

Ex 6: $\int_{-1}^{2} (3 - 2x)\,dx$

Ex 7: $\int_{-3}^{0} (1 + \sqrt{9 - x^2})\,dx$

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The midpoint Rule

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\Delta x \quad \text{where } \Delta x = \frac{b-a}{n}$$

and $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$. ← the midpoint on $[x_{i-1}, x_i]$.

Ex 8: Estimate $\int_0^{\pi} \sin(x)dx$ using a Riemann sum w/ 3 subintervals. Compare left, right, and midpoint approximations.

Properties of the Definite Integral

I) $\int_a^b f(x)dx = -\int_b^a f(x)dx$ $\quad \Delta x = \frac{b-a}{n} = -\frac{a-b}{n}$

II) $\int_a^a f(x)dx = 0$

III) $\int_a^b c\, dx = c(b-a)$

IV) $\int_a^b c \cdot f(x)\, dx = c \cdot \int_a^b f(x)dx$

V) $\int_a^b [f(x) \pm g(x)]\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)dx$

Ex 9: Use the properties of integrals to evaluate

$\int_0^1 (6 - x^3)dx$ (remember $\int_0^1 x^3\, dx$ from Day 1).

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VI) $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$

p389, "Not easy to prove."

Ex 10: Write $\int_{-2}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx - \int_{-2}^{-1} f(x)\,dx$

as a single integral.

comparison properties of the integral

VII) If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$

VIII) If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

IX) If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

If time, prove (VIII)

Ex 11: Use IX to estimate $\int_1^2 \frac{dx}{x}$.