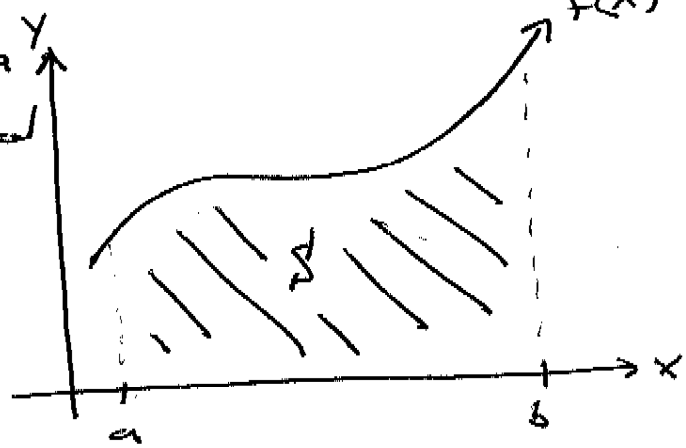
Our goal in math 125 is to find the area "under" an arbitrary curve $f(x)$.

In general, find the area of the region S bounded by the continuous curve $y=f(x)$ $[f \geq 0]$, the x-axis, and the lines $x=a$, and $x=b$.

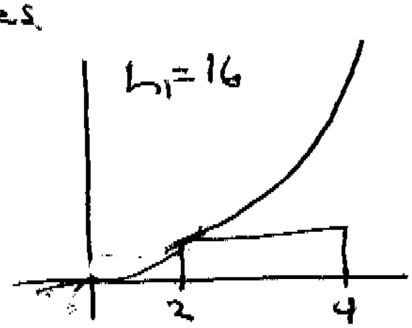


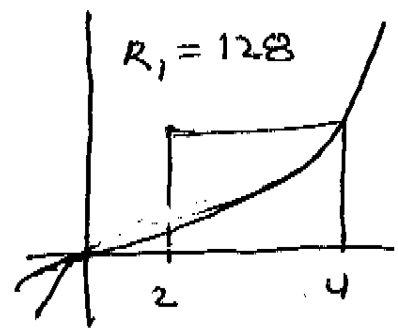
We will approximate the area of S w/ rectangles.

<u>Ex 1:</u> Consider the region S under $y=x^3$ on $2 \leq x \leq 4$. Approximate the area A of S using rectangles.

① 1 rectangle.

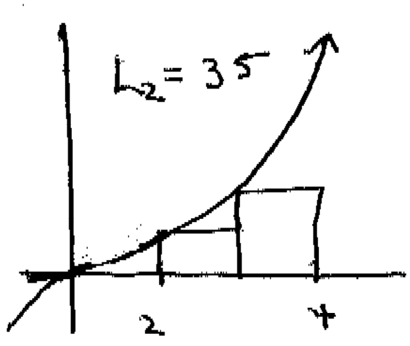
$16 < A < 128$

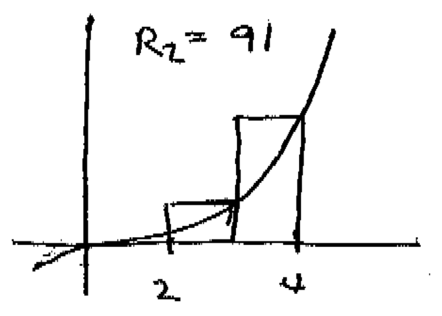




② 2 rectangles.

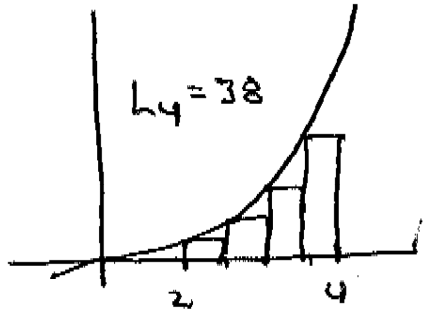
$35 < A < 91$

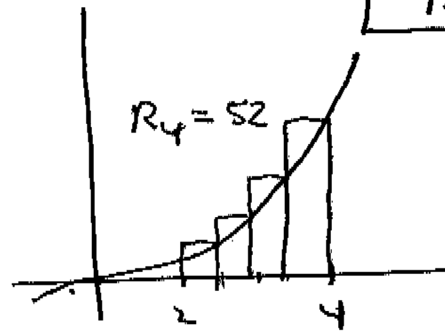




③ 4 rectangles,

$38 < A < 52$





Notice that $\lim_{n\to\infty} L_n = A$ and $\lim_{n\to\infty} R_n = A$.

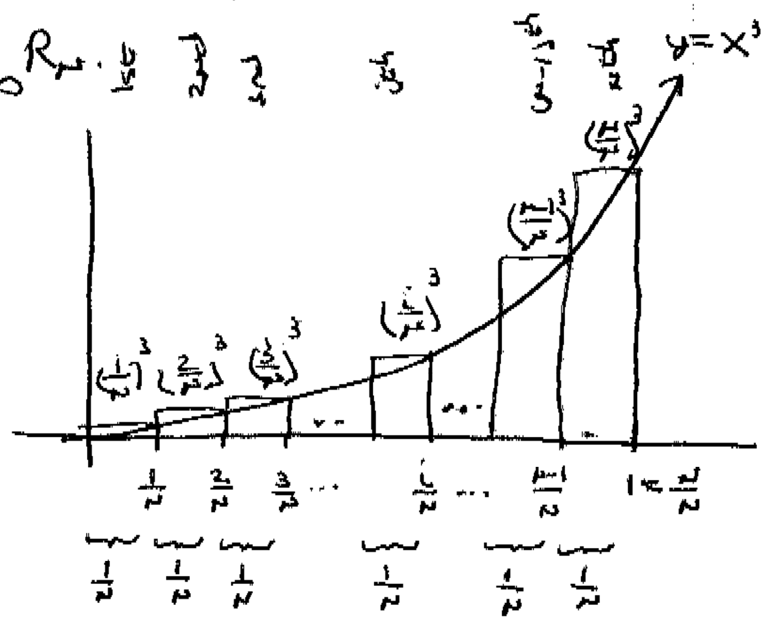
So, what we want is to find R_n (we could choose L_n) for an arbitrary value of n and then let $n \to \infty$.

In order to simplify calculation, consider the following example.

<u>Ex 2:</u> Find the area under $y=x^3$ from $x=0$ to $x=1$ using rectangles w/ right end points and n rectangles w/ equal width. Then calculate $A = \lim_{n\to\infty} R_n$.

$$R_n = \left(\frac{1}{n}\right)^3 \cdot \frac{1}{n} + \left(\frac{2}{n}\right)^3 \cdot \frac{1}{n} + \left(\frac{3}{n}\right)^3 \frac{1}{n} + \dots + \left(\frac{n}{n}\right)^3 \cdot \frac{1}{n}$$

$$= \frac{1}{n^4}\left(1^3 + 2^3 + 3^3 + \dots + n^3\right)$$



5.1
3/4

Now, we all know that

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{1}{4} n^2 (n+1)^2$$

$$= \frac{1}{4} (n^4 + 2n^3 + n^2)$$

so $R_n = \frac{1}{n^4} \cdot \frac{1}{4} (n^4 + 2n^3 + n^2)$

$$= \frac{1}{4} \left(1 + \frac{2}{n} + \frac{1}{n^2}\right)$$

AND $A = \lim_{n \to \infty} R_n$

$$= \lim_{n \to \infty} \frac{1}{4} \left(1 + \frac{2}{n} + \frac{1}{n^2}\right)$$

$$= \frac{1}{4}$$

conclusion, we have shown that the exact area under $y = x^3$ on $0 \le x \le 1$ is $1/4$. We would have reached the same conclusion using rectangles w/ left end points.

NOTATION

a) $1^3 + 2^3 + 3^3 + \dots + n^3 = \sum_{i=1}^{n} i^3$

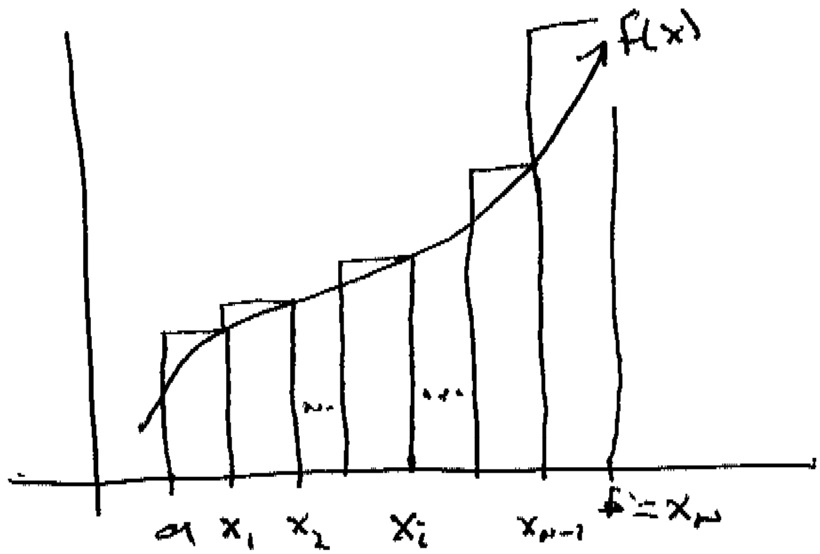
b) $\sum_{i=1}^{10} \frac{i-1}{i}$

c) $x_0 + x_1 + x_2 + \dots + x_{n-1} = \sum_{i=0}^{n-1} x_i = \sum_{i=1}^{n} x_{i-1}$

5.1
4/4

Generallizing the procedure of ex2 and using the sigma notation:

To find the area under $y=f(x)$ from $x=a$ to $x=b$ using n rectangles of equal width...



Let $\Delta x = \frac{b-a}{n}$

and $a = x_0$

$b = x_n$

so $x_1 = a + \Delta x$

$x_2 = a + 2\Delta x$

$x_i = a + i\Delta x$

$\vdots$

right endpoints

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_i)\Delta x + \dots + f(x_n)\Delta x$$

$$= \sum_{i=1}^{n} f(x_i)\Delta x$$

and $A = \lim\limits_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

left endpoints

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

In general

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \quad \text{where } x_{i-1} \le x_i^* \le x_i$$

for $i \in \{1, 2, \dots, n\}$