Math of Finance
Math 111: College Algebra
Academic Systems

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Contents

Introduction

This handout will introduce some concepts related to the exponential growth of money. This is particularly relevant to students with business related majors as well as the average participant in today's world of personal finance involving loans, savings, investments and annuities.

Since this is a mathematics course, the mathematical principles underlying the concepts will be presented. The authors of this material cannot claim to be business/finance specialists, so the problems presented are not as complex as you would find out in the "*real world*". Since this is an introductory treatment, only a selection of concepts will be covered. If you are majoring in subject matter that uses this type of material significantly, I'm sure your later courses will go much more in depth.

That said I will try to introduce you to a few of the basics in personal finance that I do feel comfortable speaking to. Let's begin with a check-up.

(0.1) Personal Finance: How healthy are you financially?

Where do you stand financially? Compare your debts to your earnings and find out. The debt-to-income is one of the most straightforward ways to determine your financial health.

a.) Collect credit billing statements to get a good estimate of what you owe each month.

b.) Make an outline of bills (car loan, student loans, mortgage, credit card, and rent) and the amount you pay each month. Do not include taxes and utilities.

c.) Calculate your monthly income, before taxes, including additional income (if applicable) for allowances, investments, or child support.

d.) Divide your payments from (1.) by your monthly income from (3.). If your income is $2000 per month and you make loan payments of $700, your debt-to-income ratio is 35% ($700/$2000 = 0.35).

What does it mean?

- *36% or less*
  This is where you want to be. Many lenders take this into consideration when you apply for a loan.
- *37% to 42%*
  Start paying your debts down. You may be headed for financial difficulties.
- *43% to 49%*
  This is a high debt-to-income ratio. You need to take immediate action to take care of debts.
- *Above 50%*
  You should seek professional help to help reduce your debts.

From a newsletter distributed by the Boeing Employees Credit Union

I will develop formulas that cover basic savings plans and repayment of loans. Though challenging, understanding how these formulas are reached might be required to do some of the more advanced homework problems. You will be required to "memorize/know" the formulas - at least to the extent of being able to match each formula with its name.

Essentially, the first three sections cover the material in Academic Systems Lesson 17.1. The following sections use this mathematical foundation to build an understanding of compounding, annuities, and loans.

Complete solutions to a few of the problems are given in the appendix at the back, as well as some hints other their solutions. Answers will be given to many of the odd numbered problems. Be sure you show your use of the appropriate formulas that lead to your solutions.

Section 1
Sequences

To understand the concepts to be presented later, I need to introduce you to the concept of a **sequence**. (This may be review--but bear with me as I try to make this material stand on its own.)

Definition: A **sequence** is a function *f* whose domain (inputs) is {1, 2, 3, 4, . . .}. The outputs $f(1), f(2), f(3), \ldots$ are the **terms** of the sequence, with $f(1)$ the first term, $f(2)$ the second term, etc.

Example (1.1): If you were told that $f(x) = 3x\text{-}5$ was to represent a sequence, you would immediately know that x was only to take on values of 1, 2, 3, . . . etc. The terms of this sequence would be -2 (by letting $x = 1$), 1 (by letting $x = 2$), 4 (by letting $x = 3$), etc. In fact if you calculated a few more terms for higher values of x, you would find the terms of this sequence are as follows:

-2, 1, 4, 7, 10, 13, 16, 19, . . .

Since *f* is a sequence you would not let $x = .5$, nor $x = -7.2$, nor any other values you might normally consider if *f* was a function whose domain was all real numbers.

NOTATION: Subscripts are most often used in describing the terms of a sequence rather than standard function notation. f_1, f_2, f_3, etc. instead of $f(1), f(2), f(3)$, etc. Usually letters at the beginning of the alphabet are used to name a sequence function, rather than *f*, *g*, or *h*. Also the letter n is commonly used for the domain variable rather than x. Thus instead of $f(x) = 2x$, it would be common to see the sequence described by $a_n = 2n$.

The use of this alternate notation should alert you that the function is a sequence.

There are countless examples of sequences - just think up any function that will accept the natural numbers 1, 2, 3 . . . as inputs! The outputs are usually displayed in order, separated by commas. It's the outputs that are normally considered to be the sequence.

Example (1.2): Let $a_n = 3n - 1$. The first 6 terms of this sequence are: 2, 5, 8, 11, 14, and 17. The 100th term is 299. (just let $n = 100$) Note that what you see are the outputs of the function and the **order** in which you see the outputs tell you what the inputs were (i.e. 1st number indicates $n = 1$; second number means $n = 2$; third number indicates $n = 3$; etc.).

Example (1.3): Let $b_n = n^2 + n$. The first 6 terms of this sequence are 2, 6, 12, 20, 30, 42. The 80th term is 6480. (Check me out using your calculator with $n = 80$).

While there are many classes of sequences, our primary interest (in algebra) lies in arithmetic and geometric sequences.

The Problem Set

In finding sums and terms, show that you're using formulas rather than just simply doing all the work on your calculator. Of course, a calculator double-check is a fun way to check to see if your theory is on the mark.

1. Find the values of the first 6 terms of these sequences:

(a) $a_n = 3 + 2n^2$
(b) $b_n = n\,(n + 2)$
(c) $c_n = n^n$
(d) $f_n = (-1)^{n-1}\dfrac{n+1}{n^2}$

2. For each of the following sequences, (i) give the next 3 terms of the sequence and (ii) give a function definition of the sequence.

(a) The sequence a_n starts as 1, 4, 9, 16, 25, . . .

(b) The sequence f_n starts as 1/2, 2/3, 3/4, 4/5, . . .

(c) The sequence d_n starts as 3, 8, 13, 18, 23, ...

Section 2
Arithmetic Sequences and Series

One special type of sequence is the arithmetic sequence. A few examples include:

Example (2.1) 1, 4, 7, 10, …	**Example (2.2)** 10, 8, 6, 4, …
Example (2.3) $\frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, ...$	**Example (2.4)** -9, -5, -1, 3, …

What do these sequences have in common? Well, in each case the sequence is obtained by adding a fixed constant to the previous term (sometimes the constant is negative as in **(2.2)**). What should we call this constant?

Consider what happens when we find the difference between subsequent terms – say in example **(2.1)**.

$$\begin{aligned} 4-1&=3 \\ 7-4&=3 \\ 10-7&=3 \end{aligned}$$

Since subtraction of subsequent terms always results in the same number, we call this number the <u>common difference</u>.

If you know the common difference *d* and one element, say a_1, then:

$$\begin{aligned} &\text{1st term is:} && a_1 = a_1 \\ &\text{2nd term is:} && a_2 = a_1 + d \\ &\text{3rd term is:} && a_3 = a_2 + d \\ &\vdots \\ &n^{th} \text{ term is:} && a_n = a_{n-1} + d \end{aligned}$$

Working from the last line, we can subtract a_{n-1} from both sides of the equation $a_n = a_{n-1} + d$ and "discover" that $d = a_n - a_{n-1}$. Once again justifying *d*'s title as the common difference.

Example (2.5): Find the 10^{th} term in the arithmetic sequence where $a_1 = -3$ and $d = 2$.

Solution: We start with a_1 and add *d* to find the subsequent element: $a_2 = -3 + 2 = -1$. To find a_3 we repeat the same process: $a_3 = -1 + 2 = 1$. Continuing in the same manner, we find the sequence -3, -1, 1, 3, 5, 7, 9, 11, 13, 15 and so $a_{10} = 15$

That wasn't too bad, but imagine the difficulty in finding a_{100} and a_{1000} in this manner. There must be a better way. Let's consider a pattern.

$$
\begin{aligned}
a_1 &= a_1 + 0d && = a_1 + (\mathbf{1}-1)d \\
a_2 &= a_1 + d = a_1 + 1d && = a_1 + (\mathbf{2}-1)d \\
a_3 &= a_2 + d = (a_1 + d) + d = a_1 + 2d && = a_1 + (\mathbf{3}-1)d \\
a_4 &= a_3 + d = (a_1 + 2d) + d = a_1 + 3d && = a_1 + (\mathbf{4}-1)d \\
&\vdots && \vdots \\
a_n &= a_{n-1} + d = [a_1 + (n-2)d] + d && = a_1 + (\boldsymbol{n}-1)d
\end{aligned}
$$

So, we now have the following three formulas to use when working with arithmetic sequences:

$$a_n = a_{n-1} + d \qquad (2.6)$$

$$d = a_n - a_{n-1} \qquad (2.7)$$

$$a_n = a_1 + (n-1)d \qquad (2.8)$$

Example (2.9): Find the 37th term of the arithmetic sequence 4, 1, -2, -5, …

Solution:
Step 1: Find a_1 and *d*. Notice that the first term is $a_1 = 4$. Calculate *d* using formula (2.7) (or observe) that $d = -3$. *d* is negative because the terms are decreasing.
Step 2: Use formula (2.8) to find $a_{37} = 4 + (37-1)(-3) = 4 + 36(-3) = -104$

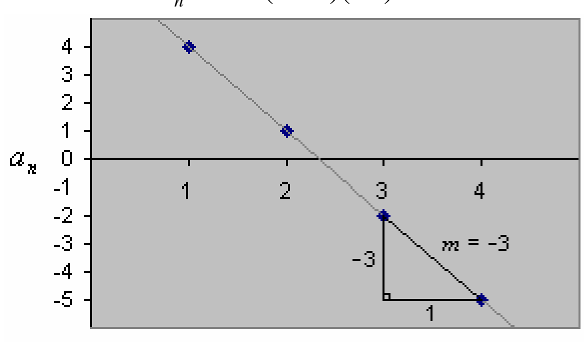
Example (2.10): Find the *n*th term of the arithmetic sequence 4, 1, -2, -5, …

Solution: Another way to find the n^{th} term in an arithmetic sequence is to graph the points (n, a_n). These points fall on a line whose slope is *d* and whose *y*-intercept is $a_1 - d$

The graph shows the points along with the line that intersects them. Choosing two points, we find that $d = -3$. Now select a point, say $(1, 4)$, and substitute these values into the formula $y = mx + b$ (the slope-intercept form of a line). From this, we have

$$
\begin{aligned}
4 &= -3(1) + b \\
b &= 7
\end{aligned}
$$

Solving for a_1 in the formula $b = a_1 - d$, we have that $a_1 = b + d$ which in our case means that $a_1 = 7 + (-3) = 4$. Thus, the n^{th} term is $a_n = 4 + (n-1)(-3)$.



Example (2.11): Find the 53[rd] term of the arithmetic sequence where $a_{13} = 17$ and $a_{24} = 49$

Solution:
Step 1: Find a_1 and *d*. To do this, we must construct a system of two linear equations with two unknowns. We find these equations by substituting the known values into equation (2.8).

For *n* = 13: $17 = a_1 + 12d$ and for *n* = 24: $49 = a_1 + 23d$. We can solve this system using a number of methods. Here we will use the elimination method (also called addition).

$$\begin{cases} a_1 + 12d = 17 \\ a_1 + 23d = 49 \end{cases} \tag{2.12}$$

Subtract the second line from the first in (2.12) resulting in the equation:

$$-11d = -22 \tag{2.13}$$

Divide both sides of (2.13) by -11 to determine that *d* = 2. Substitute this result back into line 1 of (2.12) to form the equation:

$$\begin{aligned} a_1 + 12(2) &= 17 \\ a_1 + 24 &= 17 \\ a_1 &= -7 \end{aligned} \tag{2.14}$$

Step 2: Use formula **(2.8)** to find $a_{53} = -7 + (53-1)(2) = -7 + 52(2) = 97$.

So far, we have considered sequences or lists of numbers. Now, we will focus our attention on adding up the terms in a sequence. We often call the sum of a sequence a <u>series</u>. Specifically, let us find the sum of the arithmetic series. However, no exposition on the arithmetic series would be complete without the following entertaining story about the great mathematician Carl Gauss

Carl Gauss (1777-1855): A Historical Note

Carl Friedrich Gauss, who was born in 1777 in Braunschweig, Germany, the son of a masonry foreman, was a master at exposing unsuspected connections. He was a mathematical prodigy, and in his old age he liked to tell stories of his childhood triumphs.

At the age of ten, he was a show-off in arithmetic class at St Catherine elementary school, "a squalid relic of the Middle Ages... run by a virile brute, one Buttner, whose idea of teaching the hundred or so boys in his charge was to thrash them into such a state of stupidity that they forgot their own names."

One day, as Buttner paced the room, rattan cane in hand, he asked the boys to find the sum of all the whole numbers from 1 to 100. The student who solved the problem first was supposed to go and lay his slate on Buttner's desk; the next to solve it would lay his slate on top of the first slate; and so on.

Buttner thought the problem would preoccupy the class, but after a few seconds Gauss rushed up, tossed his slate on the desk, and returned to his seat. Buttner eyed him scornfully, as Gauss sat there quietly for the next hour while his classmates completed their calculations.

As Buttner turned over the slates, he saw one wrong answer after another, and his cane grew warm from constant use. Finally he came to Gauss's slate, on which was written a single number: 5,050, with no supporting arithmetic.

Astonished, Buttner asked Gauss how he did it. When Gauss explained it to him, the teacher realized that this was the most important event in his life and from then on worked with Gauss always, plying him with textbooks, for which Gauss was grateful all his life.

www.cannovan.com/puzzles/gauss.htm

So, how did Gauss solve the problem? Well, we will never know exactly what went on in his mind. However, perhaps he solved it by using the following logic.

Example (2.15): Find the sum of the series 1 + 2 + 3 + … + 99 + 100.

***Solution*:** Consider the series

$$1+2+3+...+98+99+100 \tag{2.16}$$

Reversing the terms does not change sum

$$100+99+98+...+3+2+1 \tag{2.17}$$

Now we will add (2.16) and (2.17) together

$$\begin{array}{ccccccccccccc} 1 & + & 2 & + & 3 & + & \cdots & + & 98 & + & 99 & + & 100 \\ + & & + & & + & & & & + & & + & & + \\ 100 & + & 99 & + & 98 & + & \cdots & + & 3 & + & 2 & + & 1 \\ \hline 101 & + & 101 & + & 101 & + & \cdots & + & 101 & + & 101 & + & 101 \end{array} \tag{2.18}$$

How many terms of 101 are we adding? That is right, 100 terms are being added. So our new sum is 100(101). However, this sum isn't identical to the original because it contains (2.16) twice. So, we divide our result by 2 to get the final answer.

$$\frac{100(101)}{2} = 5050 \tag{2.19}$$

More generally, $\frac{100(101)}{2} = \frac{100(1+100)}{2} = 5050$

So, what was Gauss' trick? Well, we presume that Gauss added the first and last terms, multiplied the sum by the total number of terms, and divided this result by two. Very cool!

Theorem: The Sum of an Arithmetic Series.

$$S_n = \frac{(a_1 + a_n)n}{2} \tag{2.20}$$

We'll use the notation S_n to indicate the **sum** of n terms of the arithmetic sequence.

Example (2.21): Find the sum of the series 17 + 28 + 39 + … + 1502 + 1513

***Solution*:** This is an arithmetic series, so all we need is the first term, the last term, and the total number of terms n in the series. We are given $a_1 = 17$ and $a_n = 1513$; only n requires work to find. By inspection, we can see that $d = 11$. Using formula (2.8), with n as the only unknown, we have that:

$$\begin{aligned}
&1513 = 17 + (n-1)(11)\\
&1513 = 17 + 11n - 11\\
&1513 = 6 + 11n\\
&1507 = 11n\\
&\underline{n = 137}
\end{aligned}$$

But now that we have n, the sum is within our grasp using formula (2.20).

$$S_{137} = \frac{(17 + 1513)(137)}{2} = 104805$$

Example (2.22): Add the 3rd term through the 103rd term in the series where $a_n = -5 + 2n$.

***Solution*:** Sometimes we are at a loss as to how to proceed. At such times, looking for a pattern can be advantageous. To find the pattern, let's start plugging in values for n. This time, we will begin with $n = 3$. Doing this we arrive at the series:

$$1 + 3 + 5 + ... + 197 + 199 + 201 \tag{2.23}$$

Wait a second! That looks just like Gauss' problem. So, we could use the trick used in example **(2.15)** of reversing the sum or we could simply observe that the first term is 1, the last is 201, and there are . . . how many terms? How many integers are there between 3 and 103 <u>including the end values</u>? You should be able to convince yourself there are 101 terms, so $n = 101$.

Once again, we apply formula (2.20), and are left with:

$$S = \frac{(1 + 201)(101)}{2} = 10201$$

You may have noticed that we did not use S_{101} or S_{103}, but used S in the preceding example. This was to avoid subscript confusion. Remember, the mathematical language is designed for clarity. Gauss could solve the problem without subscripts and so can we. Often times the subscripts hide the elegance of the solution.

The Problem Set

In finding sums and terms, show that you're using formulas rather than just simply doing all the work on your calculator. Of course, a calculator double-check is a fun way to check to see if your theory is on the mark.

1. Find the forty-ninth term, a_{49} of the arithmetic sequence 7, 4, 1, …

2. Find the first term, a_1, and the common difference, *d*, of the arithmetic sequence whose third term, a_3, is 16 and whose fifteenth term, a_{15}, is -8.

3. For the arithmetic sequence 18, 13, 8, 3, …

   a. Find the common difference *d*.
   b. Find the twenty-fourth term a_{24}.
   c. Find the sum, S_{36}, of the first 36 terms.

4. Find the first term, a_1, and common difference, *d*, of the arithmetic sequence whose sum of the first 12 terms, S_{12}, is 246 and whose twelfth term, a_{12}, is 37.

5. For the arithmetic sequence 3, 7/2, 4, 9/2, …

   a. Find the common difference *d*.
   b. Find the eleventh term, a_{11}.
   c. Find the sum, S_{50}, of the first 50 terms.

6. A ball rolling down an inclined plane moves 8 feet the first second. In each second thereafter it moves 16 feet more than in the preceding second.

   a. How far will the ball move during the tenth second?
   b. How far will it have moved during the first 10 seconds?

7. Find the sum of the first 10,000 terms of the arithmetic sequence whose tenth term, a_{10}, is -11 and whose nineteenth term, a_{19}, is -71.

Section 3
Geometric Sequences and Series

Example (3.1): Let $c_n = 3(2^n)$. The first 6 terms of this sequence are:

6, 12, 24, 48, 96, 192.

The 60th term is 3,458,764,513,820,540,928. WOW! (I used a TI-92 - you can't get this exact value on a normal calculator. A normal scientific calculator will change to scientific notation and give you 3.458764514 x 10^{18} - which is good enough for government work!)

Example (3.2): Let $a_n = 2(3^{n-1})$. The first 5 terms are 2, 6, 18, 54, and 162. The 60th term is 28,260,772,183,477,469,009,529,622,134. (Want to do that by "hand?" Again, a normal calculator would give you 2.826077218 x 10^{28})

You might notice that examples above are exponential functions of the type:

$$y = c \cdot b^x,\ b > 0 \text{ and } b \neq 0 \text{ where } c \in \mathbb{R}.$$ (See Lesson 12.1 in Academic Systems.)

Exponential functions give us what are called **geometric sequences.**

Definition: A **geometric sequence** is of the form: $a_n = a_1 r^{n-1}$.

Note that when $n = 1$, $a_1 = a_1(r^{1-1}) = a_1(r^0) = a_1(1) = a_1$. The value of r is called the **common ratio**. (If you look at example **(3.2)** above, the *ratio* of a term compared to the previous term is always 3.)

The **ratio *r*** of a geometric sequence is $\frac{a_n}{a_{n-1}}$ for any two consecutive terms a_n and a_{n-1}. Thus $a_n = r \cdot a_{n-1}$. Hence multiply a term by r to get the next term. Since we are repeatedly multiplying by r, each term involves powers of r, as in our definition of a geometric sequence above.

From a practical point of view, to construct a geometric sequence start with a number (called a_1), then multiply by r, then multiply by r again, multiply by r, etc. To construct the sequence in example **(3.2)**, start with 2, multiply by 3 to get 6, multiply by 3 to get 18, then by 3 to get 54, etc.

Example (3.3): Construct a geometric sequence whose first term is 4 and has a common ratio of 5 and then give the formula for this sequence.

Solution: Start with the value of 4, then continue to multiply by the common ratio of 5. If you do this you will get 4, 20, 100, 500, 2500, 12500, etc.

The formula for this sequence is $a_n = 4(5^{n-1})$

You should check out that this formula gives you the terms of the sequence by trying $n = 1, 2, 3, 4, \ldots$

Example (3.4): Give a formula for the geometric sequence:

$$2, 1, 1/2, 1/4, 1/8, \ldots$$

Solution: The first term is 2, so $a_1 = 2$. We need to determine r. Can you see what number is being used as a multiplier? Sometimes it's very obvious. If not, take any term and divide by the previous term. (like $1/2 \div 1$ or $1/8 \div 1/4$) For the sequence, this always gives you an answer of 1/2.

So, $r = ½$ and $a_n = 2\,(1/2)^{n-1}$

Example (3.5): Inflation over the past 10 years has been about 3.4% per year. This means the price of something will increase by about 3.4% per year. If a loaf of bread in 1990 was \$0.95, about what would that loaf of bread be worth in 2002?

***Solution*:** If P is the price at any year, then $P + .034P$ is the new price the next year. This is equal to $1.034P$. In other words, we just simply multiply the "old price" by 1.034 to get the next "new price". Thus the next year, we have

$$1.034(1.034P) = 1.034^2\,P, \text{ etc.}$$

We generate a geometric sequence $P, 1.034P, 1.034^2\,P, 1.034^3P, \ldots, 1.034^{12}P$ with a ratio of 1.034.

If $P = \$0.95$, then after 12 years, $\$0.95(1.034)^{12} \approx \$1.42$ will be the selling price of the loaf of bread.

Now, let's see if we can find a formula that gives the **sum** of a finite geometric sequence.

Let's look at a few examples to see if we might guess at what the formula might be.

$$1 + 2 + 4 + 8 + 16 = ??? \qquad (a_n = 1\,(2)^{n-1},\ n = 1, 2, 3, 4, 5, a_1 = 1,\ \text{and } r = 2\ )$$

If you add this up with your calculator you get 31. Hmmm . . . that's simply 1 less than the next term. Let's try a few more terms: $1 + 2 + 4 + \ldots + 512$. If you add this up with your calculator, you will get 1023. You might notice that this is simply 1 less than the next term in the original sequence.

But before we jump to any conclusions, maybe we'd better try a more complicated example.

$2 + 8 + 32 + 128 + 512$ ($a_1 = 2$ and $r = 4$). This sequence adds to 682.

Now, the next term would be 2048 - Whoops - our sum is NOT the next term minus 1. You might notice that the sum of 682 is almost 1/3 of 2048. In fact, if we subtract 2 from 486 and divide by 3 we get the right answer for the sum.

$$\left(\frac{2048-2}{3} \right)$$ Hmmm.

You might check that for this sequence, if you're adding the 1st n terms, just take what would be the next term, subtract 2, (Hmmm - 2 is the first term) then divide by 3 (Hmmm again . . . 3 is one less than the ratio). This seems to give you the right answer for the sum - try it!

Do you think you see a pattern? If you think you do, try to find the sum of the first 25 terms of the sequence:

$$a_n = 4(3)^{n\text{-}1}$$

OK, check your guess with the following theorem:

Theorem: The Sum of a Geometric Series.

$$\begin{aligned} S_n &= a_1 + a_1 r + a_1 r^2 + a_1 r^3 + ... + a_1 r^{n-1} \\ &= \frac{a_1 r^n - a_1}{r-1} \\ &= \frac{\text{next term} - \text{first term}}{\text{ratio} - 1} \end{aligned}$$

We'll use the notation S_n to indicate the **sum** of n terms of the geometric sequence.

In everyday language, to add up terms of a geometric sequence, simply take the value of what would be the next term, subtract the first term, then divide by 1 less than the ratio. Note that this works for each of the examples given above. See your Academic Systems text for a more formal proof in the subsection titled "The Derivation of the Formula $S_n = \frac{a_1(1-r^n)}{1-r}$" toward the end of Lesson 17.1.

Example (3.6): Find 5 + 15 + 45 + 135 + 405 by this algorithm.

***Solution*:** The next term would be 1215 and $r = 3$. So $S_5 = \frac{1215-5}{2} = 605$.

(Check this answer by actually adding up the series with your calculator).

Example (3.7): Find the sum of the first ten terms of $a_n = 3(2^{n\text{-}1})$.

***Solution*:** $S_{10} = \frac{3 \cdot 2^{10} - 3}{2-1} = \frac{3072-3}{1} = 3069$.

Check by writing out the first ten terms, then adding them up on your calculator.

Example (3.8): Find the sum of 1 + 1/2 + 1/4 + 1/8 + 1/16 + 1/32.

***Solution*:** $r = 1/2$, $a_1 = 1$, and $a_n = 1(1/2)^{n-1}$

$$S_6 = \frac{\left(\frac{1}{2}\right)^6 - 1}{\frac{1}{2} - 1} = \frac{\frac{1}{64} - 1}{-\frac{1}{2}} = \frac{-\frac{63}{64}}{-\frac{1}{2}} = \frac{63}{32}$$

Remember, start with the next term, subtract the first term, and then divide by 1 less than the ratio.

Example (3.9): You start working at a job for $800 per month. However, as you gain experience, the boss assumes you are worth more (because you work faster and more efficiently) and gives you a raise of 10% for the next month. (i.e. You'd receive $880 for the next month). Assume a raise of 10% takes place each month. How much have you earned in 1 year of work?

***Solution*:** Your pay for each month is as follows:

Month 1: $800

Month 2: $800+800(.10)=800(1.10)

(notice we just multiply the previous month's balance by 1.10)

Month 3: $800(1.10)(1.10)=800(1.10)^2$
Month 4: $800(1.10)^2(1.10) = 800(1.10)^3$

We may continue in this manner – all the while noticing that the exponent is 1 less than the month until we get to

Month 12: $800(1.10)^{11}$

What we want is $800 + 800(1.10) + 800(1.10)^2 + \ldots\ldots + 800(1.10)^{11}$

$$S_{12} = \frac{\text{next term} - \text{first term}}{\text{ratio} - 1} = \frac{800(1.10)^{12} - 800}{1.10 - 1} \approx \frac{1710.743}{.10} = \$17,107.43$$

As you can see, that 10% raise at the end of each month really adds up. Contrast the total with just 12(800) = $9600 when no raise was given.

Example (3.10): A geometric sequence has a first term of 3.25 and a ratio of 2. Find which term is equal to 53248.

***Solution*:** We need to solve $53248 = 3.25(2)^{n-1}$ for n. This is a job for logarithms! First divide by 3.25 to get $16384 = 2^{n-1}$. Then take the log of both sides to get

$$\ln(16384) = (n-1)\ln(2) \Rightarrow \frac{\ln(16384)}{\ln(2)} = n - 1 \Rightarrow 14 = n - 1 \Rightarrow n = 15.$$

So the 15th term is the one that equals 53248.

You may have noticed that the natural logarithm "ln" was used in example **(3.10)** instead of log base 2 or the common log. This purely a matter of preference – any base would suffice. However, most calculators only have the common and natural logs and some (like the TI-92) are limited to the natural log.

In any case, you can use the change of base formula:

$$\log_b(a) = \frac{\ln(a)}{\ln(b)} = \frac{\log(a)}{\log(b)} = \frac{\log_c(a)}{\log_c(b)} \qquad (3.11)$$

For a more thorough review of logarithms, see Lesson 12.2 in the Academic Systems text.

The Problem Set

In finding sums and terms, show that you're using formulas rather than just simply doing all the work on your calculator. When working with money, round off the final answer to the nearest penny. Of course, a calculator double-check is a fun way to check to see if your theory is on the mark.

1. Write out the first 6 terms of a geometric sequence whose first term is 5 and the ratio is 3. Give the formula for this sequence. Use your formula to find the 17th term. Find the sum of the first 12 terms using the formula for the sum of a geometric sequence.

2. Write out the first 6 terms of a geometric sequence whose first term is 9 and the ratio is 1/2. Give the formula for this sequence. Use this formula to find the 11th term. Find the sum of the first 8 terms using the formula for the sum of a geometric sequence.

3. Give the formula for the geometric sequence whose first term is 7 and ratio is -1/2. Find the sum of the first 20 terms of this sequence. Give an answer to the full accuracy of your calculator and also try to give an exact answer as a fraction.

4. If inflation causes the value of a house to increase about 8% per year, what would a house that is worth $120,000 today, cost in 9 years? Give a formula for the cost of the house n years from now. What is the ratio r for this geometric sequence?

5. Write the first 6 terms of a geometric sequence whose first term is -2 and whose ratio is -2. Find the sum of the first 8 terms. Find the sum of the first 29 terms.

6. A famous problem goes as follows: Bob tells his boss he will work for "pennies" every day during the month of April. On April 1st he will work for 1 cent. The next day he will work for 2 cents, the next day for 4 cents, the next day for 8 cents, etc. The boss, being greedy (but who slept through business math, readily agrees). How much does Bob earn on the last day of April? What is the total of all of Bob's wages for the month of April? (Express your final answers in dollar format - i.e. if answer was 867 cents, express answer as $8.67).

7. A $5,000 loan taken out on the first of March is being repaid to a friend (no interest is being charged) by making monthly payments of 12% of the unpaid balance. You will make your payments on the first day of the following months. Write several terms of a sequence called B that indicates the unpaid balance at the beginning of each month (right after you make the payment). Start your sequence with $B_1 = 4400$. (The balance after the first payment.) I got this by taking 12% of 5000 and figuring out what was left. Is this a geometric sequence? If so, what is the ratio? What is the unpaid balance after 1 year? (Write the formula for the unpaid balance B in terms of n, the number of months you made a payment.) Using logarithms, find the number of months it will take before the unpaid balance reaches $10; reaches $1.

Section 4
Future Value of Investments and Compounding Your Money

You probably were exposed to the concept of "simple interest" in your beginning algebra classes. In that case, you simply took the amount of money you were saving (called the **principle**) and multiplied it by the yearly interest rate and the number of years. The interest you were earning did not become part of the principle and hence the interest did not earn further interest. Thus, if you had \$200 at 6% for 12 years, the interest earned would be \$200(.06)(12) = \$144. Thus after 12 years, you would have \$200 + \$144 = \$344.

Periodic Compounding

However, when money is compounded, you calculate the interest for a "short" period of time and add that to the principle. This forms a "new principle" that earns more interest than in the previous period. If P is the original principle, then "New Principle" $= P + Pi = P(1+i)$, where i is the interest rate per compounding period. Note, we just take the principle (what ever it may be at the time, and multiply it by $(1+i)$ to get the new principle at the next stage of compounding. If you continue to multiply by $(1+i)$ over and over, you get the formula given by

Periodic Compounding (4.1)	where	A = amount accumulated (**Future Value**)
	and	m = total number compounding periods
$A = P(1+i)^m$	and	i = interest rate per compounding period
	and	P = the original principal (**Present Value**)

Important Note: *i* versus *r*

If n = the number of times money is compounded per year, and r is the **yearly** (**nominal**) interest rate, then

$$i = \frac{r}{n}$$

For example, if your credit card's yearly interest rate is 18%, the monthly interest rate would be 1.5%.

(4.2) Types of Interest

Banks traditionally pay interest compounded monthly or daily. However, there are other places where you can earn interest that is compounded over much longer periods. Treasury bonds pay interest every six months and most companies pay dividends on a quarterly basis.

So, what is a dividend? A dividend is when a company gives some of its earnings back to the shareholder. Unlike interest from the bank which is determined (roughly) by the Federal Reserve, dividends are determined by a company's board of directors.

The catch – the yield (dividend/share price) is subject to the share price. This means that the yield is changing along with the share price.

(4.3) A Case Study in Dividends: General Electric.

Take General Electric (GE) as an example of a dividend paying stock. GE has paid a dividend each quarter for over one hundred years. In addition, GE's dividends have been raised for 27 consecutive years. Does this mean that the "interest" rate an investor earns in GE has increased each year for longer than many of us have been alive? (You can answer this question).

So, what is GE's current yield ("interest")? On August 29, 2003, shares of GE sold for \$29.40 each. GE currently pays an annual dividend of \$0.76 per share. So, the yield is \$0.76/\$29.40 = 0.0259 or 2.59%. By way of comparison, my bank pays about 1% interest.

Often the words "quarterly", "monthly", "daily" are used to indicate how often we compound money. Compounding quarterly means 4 times per year; compounding daily means 365 times per year (some books might consider $n = 360$, rather than 365 - makes only a difference of a few pennies).

Important Note: Many sources give the general compounding formula as

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

where n is the number of compounding periods per year and r is the nominal (yearly) interest rate. This is equivalent to formula **(4.1)**, since $m = nt$ would be the total number of compounding periods in t years and r/n would be i, the interest rate per period.

Let me illustrate the previous comments by doing a compound interest problem, using the basic concept of compounding, then comparing it with using the compound interest formula.

Example (4.4): Suppose \$8,000 is compounded quarterly for two years. Suppose the yearly interest rate is 6%.

Solution: Compounding quarterly, means that the interest rate is $\frac{6\%}{4} = 1.5\%$, that is $n = 4$ and $i = 0.15$. Since we are compounding 4 times per year for two years, we will have a total of 8 compounding periods (i.e., $m = 8$).

So now, let's step by step show the new principle for each successive compounding period.

Period 1: 8000+8000(.015) = \$8,120
Period 2: 8120+8120(.015) =\$8,241.80
Period 3: 8241.80+8241.80(.015) = \$8,365.427
Period 4: 8365.427 + 8365.427 (.015) = \$8490.908405

"Full accuracy" is important!

Period 5: 8490.908405 + 8490.908405(.015) = \$8,618.272031
Period 6: 8618.272031 + 8618.272031(.015) = \$8,747.546112
Period 7: 8747.546112 + 8747.546112(.015) = \$8,878.759303
Period 8: 8878.759303 + 8878.759303(.015) = \$9,011.940693

So, after two years, we have a future value of \$9,011.94. (Now we'll round to cents). Using the formula, $A = 8000(1 + .015)^8$ = \$9,011.94 (Wow - same answer!). By contrast, if your money was earning simple interest, you would have had the following amount of money:

$$A = 8000 + 8000(.06)(2) = \$8,960.$$

Example (4.5): If we invest \$5,000 by putting into a savings account paying 6% yearly interest, compounded monthly, find the future value of that money in 12 years.

Solution: $P = 5000$. Since we're compounding monthly (12 times per year), we have that $i = \frac{0.06}{12}$. Since we're compounding 12 times per year for 12 years, that's a total of 144 compounding periods. Thus $m = 144$.

$$\text{So, } A = 5000\left(1+\frac{0.06}{12}\right)^{144} = \$10,253.75 \text{ (rounded to nearest penny).}$$

Warning: Be sure to use the full accuracy of your calculator. Remember to only round off your results in your **final answer** to the nearest penny. You should learn to use your calculator well enough so that you basically never have to re-enter intermediate results by hand.

Example (4.6): If the future value of an investment after 10 years is to be \$5,500 where the money was compounded quarterly at 7.7% nominal interest, what is the present value?

Solution: In this case, we are to solve for the original principle that was invested. Quarterly compounding implies $i = \frac{.077}{4}$ (compounded 4 times per year). So, we must solve $5,500 = P\left(1+\frac{0.077}{4}\right)^{40}$. (Did you catch that $m = 4(10) = 40$?) Simply divide both sides by $\left(1+\frac{0.077}{4}\right)^{40}$ and get:

$$P = \frac{5500}{\left(1 + \frac{0.077}{4}\right)^{40}} = \$2,565.27$$

Example (4.7): If you invest \$2,500, compounded monthly at an annual rate of 5.5%, how much would you have after 15 months? After 30 years? (Always round off final answer to nearest penny).

Solution: Let m = 15. Since there are 12 compounding periods per year, we have that $i = \frac{.055}{12}$.

$A = \$2500\left(1+\frac{0.055}{12}\right)^{15} = \$2,677.50$. If there are 12 compounding periods per year, after 30 years there'd be 12(30) (which equals 360) compounding periods, hence $m = 360$ and you'll get \$12,968.47.

Now that we understand periodic compounding and future value, let's move on to continuous compounding – a special case of periodic compounding where we consider n to be infinitely large.

Continuous Compounding

Now, if we take the formula $A = P(1+i)^m$ and consider our compounding periods per year to be infinitely numerous (faster than once every second, or once every micro-second, etc) the formula evolves into (See Lesson 12.1 in your text):

Continuously Compounding	Where	r = nominal(yearly) interest rate
	and	t = number of years
$A = Pe^{rt}$		
		e is an irrational number $e \approx 2.71828$......

It should be noted that the number of compounding periods has no meaning in this case - in one sense, the number of compounding periods is $n = \infty$.

For the continuously compounding case, since there is no such thing as a compounding period, you may use any value you want for t as long it represents years. You do not have to have a whole number as a value for t. For example, use $t = 2.5$ for 2 1/2 years or if the time is 9 months, use ¾ for the value of t.

Example (4.8): If you invest \$2,500, compounded continuously at an annual rate of 5.5%, how much would you have after 7.5 years? After 200 months?

Solution: $A = \$2500\, e^{7.5(.055)}$ = \$3,776.47; $A = \$2500\, e^{(200/12)(.055)}$ = \$6,252.35 (the "trick" is that t must be expressed in years - thus 200 months = 200/12 yrs)

Example (4.9): Compounding more often (<u>keeping all other factors the same</u>) always results in a larger future value. Consequently, continuous compounding is the ultimate compounding strategy. Consider $P = 8000$, 10 years, 6% rate:

Solution: With increasing values for n, we have:

(a) compounded quarterly:	$A = 8000(1+.06/4)^{40}$ = \$14512.14727
(b) compounded monthly:	$A = 8000(1+.06/12)^{120}$ = \$14555.17387
(c) compounded daily:	$A = 8000(1+.06/365)^{3650}$ = \$14576.23163
(d) compounded hourly:	$A = 8000(1+.06/8760)^{87600}$ = \$14576.92049
(e) compounded continuously:	$A = 8000\, e^{10(.06)}$ = \$14576.9504

Now, let's look at a series of examples that illustrate trying to solve these compound interest equations for various variables, depending on the known information. The three examples should demonstrate the need to know a variety of equation solving techniques.

As you will see, similar questions can require different strategies for solving the resulting equation! That's why we study various techniques in your general algebra classes before we tackle some of the more sophisticated problems.

Example (4.10): Suppose you have received a large amount of money from an unexpected source. You wish to take part of that money and invest it in a long-term CD that pays 4.5% nominal interest, compounded continuously. You want the CD to be worth $70,000 in 15 years for your child's education at Stanford. How much should you invest?

Solution: You wish to find the **present value** of an investment that will have a **future value** of $70,000 in 15 years. To do this is to solve the algebra equation:

$$70{,}000 = P\, e^{.045(15)}$$

Divide both sides by $e^{.045(15)}$ and get $\dfrac{\$70{,}000}{e^{0.045(15)}} = P$. Using a calculator, you get P = $35,604.95.

Example (4.11): Suppose you wish to invest $5,000 in an account that will have a future value of $10,000 in 7 years. If you wish to have your investment compounded continuously, what nominal interest rate would be required?

Solution: Solve $10000 = 5000\, e^{7r}$. Ha - we need logarithms since the unknown is part of an exponent! Divide by 5000 results in:

$$2 = e^{7r} \Rightarrow \ln 2 = 7r \Rightarrow r = \frac{\ln 2}{7} = .09902 = 9.902\%$$

We need an investment with a return of 9.9% in order to double our money in 7 years.

Example (4.12): Same question as in previous example, except the investment is compounded monthly.

Solution: Since r is our nominal (yearly) rate, r/12 is the interest rate per period we need for our compounding formula.

Solve $10000 = 5000\left(1+\dfrac{r}{12}\right)^{12(7)} \Rightarrow 2 = \left(1+\dfrac{r}{12}\right)^{84}$. In this case, our "unknown" is <u>not</u> in the exponent, so logarithms are not needed. All we need to do is to take the "84th root" of both sides.

$$\text{So } \sqrt[84]{2} = 1+\frac{r}{12} \Rightarrow \sqrt[84]{2}-1 = \frac{r}{12} \Rightarrow r = 12\left(\sqrt[84]{2}-1\right).$$

Our calculator gives us r = .0994 = 9.94%. As you can see we'd need a slightly higher investment rate than in the previous example. Of all compounding choices, compounding continuously always gives us the highest return (assuming time and nominal rates are the same!).

Prior to working through the problem set, write down definitions for the following vocabulary words.

Future Value:

Present Value:

Yearly Nominal Rate:

The Problem Set

General Instructions: Show formulas used to solve problems. Display the appropriate numbers in the formulas, so partial credit can be assigned if the results aren't quite right. When working with money, round off the final answer to the nearest penny.

1. If you put a lump sum of $5,000 into an investment that pays 6%, compounded monthly, how much will you have after 15 years?

2. If the present value of a $6,000 investment pays a nominal interest rate of 7.5%, compounded continuously, what will be the future value of that investment in 20 years?

3. For a present value of $10,000 and an annual interest rate of 8%, compute the future value after 20 years for each of these compounding strategies:

a.) compounded yearly
b.) compounded quarterly
c.) compounded monthly
d.) compounded daily (365 days per year)
e.) compounded every hour
f.) compounded continuously

4. If the future value of an investment in 30 years is $150,000, what was the present value, assuming the investment was compounded daily at 5.5% yearly interest.

5. You just inherited a large sum of money. You plan on using part of it in risky investments, but you want to be sure that in case those fail, you have part of your winnings that will provide for you when you turn 60 years old in 15 years. How much should you put into a "safe" 4.5% investment, compounded monthly, that will give you $600,000 when you turn 60?

After solving the given problem, answer the same question, but use your actual age instead of assuming an age of 45 (unless, of course, you are 45).

6. Population growth is often considered as a compounding problem, compounded annually. If a population of Black Bears is 60,000 in 1995, how many bears do we estimate there will be in the year 2020 if the annual growth rate is 2.4%?

7. $5,000 is invested in an account compounded monthly at a nominal interest rate of 6%, for 10 years. At the end of that time, your money is pulled from the account and is reinvested in an account compounded continuously at a rate of 6.5%. The money is left in that second account for another 10 years. How much will you have after that time?

8. Inflation is considered to be growth with annual compounding. Suppose the inflation rate has been about 3.1% for the past 10 years. A loaf of bread that today costs $2.18, would probably have cost about how much 10 years ago?

9. Exactly 7 years ago, Betty put in $8,000 into an investment that compounded her money continuously at 7.5% annual interest rate. She then took out all of her money from the investment and used 60% of it as a down payment on a car. If she puts the balance in her credit union that compounds her money quarterly at a 5% nominal interest rate, how long before she will have \$8,000 again?

10. Answer the following questions regarding the importance of *P*.

a.) If you start with present value of \$1,500, compounded continuously at 7%, how long will it take to triple in value?
b.) If you start with a present value of \$10, compounded continuously at 7%, how long will it take to triple in value?
c.) State a conclusion based on these two examples (along with others made up to make sure you're right - change only your starting amount).

11. Suppose you start with a present value of \$800, how long will it take to double in value if you are compounding the money monthly at a yearly interest rate of 6.5%?

12. \$5,000 is put in to account A and \$6,000 into another account B. The money in account A is compounded quarterly at a nominal rate of 6.8%, whereas the money in account B is compounded daily at a nominal rate of 5%. At the end of 6 years, they are cashed out and the total is put into a single savings account paying 6% compounded continuously. 10 years later the savings is used as a down payment buy a vacation cabin that will cost \$124,000. How big of a mortgage will have to be taken out to make up the difference.

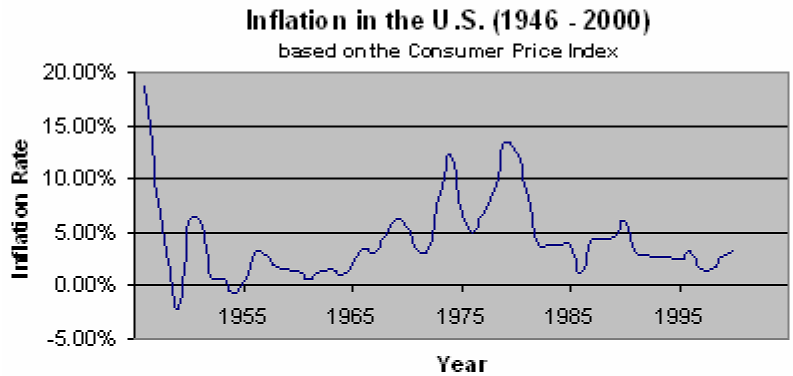
13. What interest rate will result in a future value of \$8,000, starting with a present value of \$3,500 that was compounded quarterly for 10 years?

Section 5
Effective Annual Interest Rate

When banks (or other institutions) offer many choices in savings or investments (different interest rates, different compounding periods) it's confusing as to what is the best "deal". One way to help compare these different options is to convert these to an "annual interest rate", called the **effective annual interest rate (or the *APY*; Annual Percentage Yield).** Most states require this type of disclosure from financial institutions - the name may be somewhat different, but the concept is the same.

(5.1) Historical Note: Inflation

Never underestimate the power of inflation. In the U.S., inflation has fluctuated throughout the years. However, our inflation has never rivaled that in pre-Hitler Germany (1922-23) where postage on a letter cost 2 German marks on January 2, 1922 and 100,000,000,000 marks on December 1, 1923.



It is common to calculate interest rates (after converting them into percents) to two decimal places.

To calculate the effective interest rate r_{eff} we find out the effect on a \$1 investment for 1 year:

(i) Periodic Compounding: If you're calculating the future value of \$1 for 1 year using normal compounding of n periods per year, the future value of \$1 would be:

$$\$1\left(1+\frac{r}{n}\right)^{n(1)} = \left(1+\frac{r}{n}\right)^{n}$$

where n is number of compounding periods per year.

(ii) Continuous Compounding: If compounding continuously, the \$1 becomes:

$$\$1\ e^{r(1)} = e^{r}$$

The **effective interest rate r_{eff} is the rate that would result in the same future value when you compounded the \$1 annually for one year.**

Thus the future value of the \$1 would be

$\$1(1 + r_{eff}) = 1 + r_{eff}$.

(a) $1+r_{eff} = \left(1+\frac{r}{n}\right)^{n}$

Or $r_{eff} = \left(1+\frac{r}{n}\right)^n - 1 = (1+i)^n - 1$, where $i = \frac{r}{n}$

(b) $1 + r_{eff} = e^r \Rightarrow r_{eff} = e^r - 1$

The Effective Interest Rate:

a.) Periodic Compounding: $r_{eff} = (1+i)^n - 1$, where $i = \frac{r}{n}$

b.) Continuous Compounding: $r_{eff} = e^r - 1$

Example (5.2): Find the effective annual interest rate of money that is compounded continuously at 7% interest.

Solution: $r_{eff} = e^{0.07} - 1 \approx .0725 = 7.25\%$ (2 decimal places).

Example (5.3): Find the *APY* of money that is being compounded monthly at a nominal interest rate of 7%.

Solution: $APY = \left(1+\frac{0.07}{12}\right)^{12} - 1 = 1.07229 - 1 \approx 0.07229 \approx 7.23\%$ (2 decimal places).

Note that the effective interest rate is always slightly higher than the original interest rate.[1] This is because the effective interest rate assumes you are only compounding yearly, thus to make up for this lower number of compounding periods, the interest rate has to be made higher in order to make up the difference.

Example (5.4): In the previous example, you found that the 7% rate, compounded monthly had an *APY* of 7.229%. Now, we will use both to find the future value of \$6,000 over 7 years.

Solution: Two Methods.

(a) The future value, using 7%, compounded monthly is

$$\$6000\left(1+\frac{0.07}{12}\right)^{7(12)} = \$9,779.96$$

(b) If using the *APY*, we are compounding yearly (n = 1):

$$\$6000\left(1+\frac{0.07229}{1}\right)^{7} \approx \$9,779.96$$

Note the amounts are the same. (However, if you use the slightly rounded off value of 7.23% you will be off by about 60 cents).

[1] Unless, of course, the interest is compounded annually.

The above example should convince you there is really nothing special about the number of compounding periods in and of itself; n has to be tied into the interest rate being charged too. If a bank is only compounding yearly, or quarterly, or . . . it could be just as good of a deal as the bank with continuous compounding provided the interest rate is right.

The Problem Set

General Instructions: Show formulas used to solve problems. Display the appropriate numbers in the formulas, so partial credit can be assigned if the results aren't quite right. Display interest rate results to nearest hundredth of a percent. When working with money, round off the final answer to the nearest penny.

1. Find the annual effective rate of a savings account that is advertised to be 5.6%, compounded daily.

2. Find the annual effective rate of a savings account that continuously compounds your money at 6.5%.

3. If the annual effective rate of an investment is 7.2%. What is its actual interest rate if the investment is compounded continuously?

4. If the annual effective rate of an investment that's compounded monthly is 7.2%, what is the actual interest rate?

5. If a credit union that is compounding your money continuously, advertises that the effective interest rate is 7.4%, what one-time amount do you put into an account in that credit union that will have a future value of $10,500 in 12 years? (There are a couple of ways to approach this problem; however, one-way is definitely much easier!)

6. Suppose you have some money tied up into two investments: 1/3 of it at 5%, compounded monthly and 2/3 of it at 6.5% compounded continuously. What would be a reasonable number that represents the annual effective interest rate of your total investment? (Hint: You will have to create your own math in the spirit of the definition of the effective annual interest rate). Would you expect your answer to be closer to one interest rate than the other? Which one?

7. Looking back at how we found the *APY* (annual effective interest rate), find the monthly interest rate that is equivalent to a continuous interest rate of 7%.

8. What continuous interest rate would be equivalent to a 6% interest rate, compounded quarterly?

Section 6
Annuities

Okay, let's consider a more sophisticated savings strategy to achieve a future value at some later time.

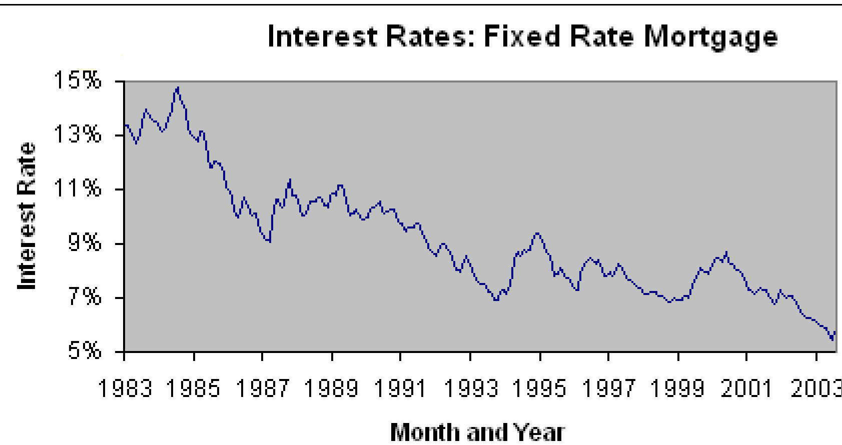
Scenario: Usually, most people don't have a big chunk of cash to set aside as savings (or an investment) at one time. Most of us might expect to achieve our savings goal by putting a smaller amount of money into the savings or investment on a regular basis over a period of time. Such a financial plan is called an **annuity.**

Note: We refer to the scenario where a fixed sum of money is paid (or distributed) on a regular basis (such as from a retirement plan) an **annuity**.

(6.1) Interest Rates

As with inflation, interest rates have changed dramatically over time. The graph given shows interest rates from January, 1983 through July, 2003.

While the problems in this text assume a constant interest rate, the world is not always that straightforward. As a consumer, you must be ready to shop around for the best rates and/or wait until you can find the rate you desire.



Data courtesy of www.hsh.com

There are many possible variations of the basic scenario and some of those variations would complicate matters in trying to come up with a simple formula. Hopefully, this class will provide you enough background that you might be able to understand an unusual situation. We will provide you an example of the most common use of this savings scenario.

Saving a fixed amount of money P at each compounding period: Suppose an amount P is put into savings. This will earn interest during the compounding period and then a payment of P is made again. This is repeated over and over. Give a formula that gives the future value at the end of t years. Let r = the annual interest rate and n = the number of compounding periods per year.

Forming the equation: Let's look at the pattern of money formed by a sequence of payments. Let $i = \frac{r}{n}$, the interest rate per period.

We start with P — P

$\downarrow$

and our first payment accumulates interest for a period, so we have — $P(1+i)$

$\downarrow$

then make the second payment P — $P(1+i)+P$

	$\downarrow$
this accumulates interest for a period	$[P(1+i)+P](1+i)$
	$\downarrow$
then we make another payment	$[P(1+i)+P](1+i)+P = P(1+i)^2 + P(1+i)+P$
	$\downarrow$
After the next payment, we have	$P(1+i)^3 + P(1+i)^2 + P(1+i)+P$
	$\downarrow$
And so on . . .	$\vdots$

What is the pattern? Just keep multiplying the previous amount accumulated by $1+i$ then add another payment *P*. Notice that in the last step, we've made 4 payments, but the largest exponent is 3 (one less than the number of payments!).

So, after *m* payments, the largest exponent in our expression above is $m-1$. So our amount accumulated is:

$$P(1+i)^{m-1} + P(1+i)^{m-2} + \ldots + P(1+i)^2 + P(1+i) + P \tag{6.2}$$

Reversing the terms in the expression, we have:

$$P + P(1+i) + P(1+i)^2 + \ldots + P(1+i)^{m-2} + P(1+i)^{m-1} \tag{6.3}$$

We see that expression **(6.3)** is a sum of a geometric sequence whose first term is *P* and whose ratio is $1+i$. If you recall our formula from the previous section, we subtract the first term from what would be the next term after the last term, and then divide by 1 less than the ratio. This would give us:

$$\frac{P(1+i)^m - P}{1+i-1} \text{ which can be simplified to } P\cdot\frac{(1+i)^m - 1}{i}$$

We will call this the Future Value, ***FV***, of this annuity. Thus, we have the formula

The Future Value of an Annuity: $FV = P\cdot\dfrac{(1+i)^m - 1}{i}$

Example (6.4): Suppose I save $100 per month toward my daughter's college education, compounded monthly, at annual rate of 5.5%. At the end of 18 years, when my daughter is ready to go to college, how much money will there be in the savings account?

Solution: $i = \dfrac{0.055}{12}$, $m = 18(12)$, $P = 100$. Putting these into the formula gives us

$$FV = 100\,\frac{\left(1+\frac{0.055}{12}\right)^{12(18)}-1}{\frac{0.055}{12}} = \$36,767.00$$

Are you impressed? If we paid $100 per month for 18 years, we made 216 payments for a total of $21,600. We almost doubled the money by making those relatively small payments over that time. Try some "what-ifs" to see how you can change the amount at the end. For example, if the payment is increased to $125, you will have $45,958.75.

Note: Keep in mind, *m* is the number of times you are adding a payment to the savings account, and *i* is the interest rate during each period of time. We are assuming the money is being compounded at the same rate at which you are adding money. The usual situation is monthly deposits and monthly compounding.

Example (6.5): Suppose Bob's company made monthly payments toward his retirement fund for 20 years. The payments were $250 and the fund earned interest at a rate of 5.6%. Bob then left the company and was allowed to "roll-over" his money in his retirement fund into a normal savings account. That savings account paid interest continuously at a rate of 6.2%. Bob chose not to make any additional payments into the savings account. 10 years after Bob left the company; Bob retired and started living off his retirement savings. How much money was in account at the time Bob retired?

Solution: We first will use our *FV* formula as in the last example, then that result will be the principal for the normal compound interest formula.

$$FV = \$250\frac{\left(1+\frac{0.056}{12}\right)^{20(12)}-1}{\frac{0.056}{12}} = \$110,190.00$$

Then $A = 110,190 \cdot e^{0.062(10)} = \$204,835.28$.

Scenario: We wish to establish an annuity for which we make periodic payments (one payment for each compounding period) in order to have a future value *FV*. Find the payment *P* that will give us that desired *FV* over a time period of *t* years. To do the task in the scenario above, just take our annuity formula and simply solve for *P*.

$$FV = P\,\frac{(1+i)^m-1}{i} \Rightarrow FV\cdot i = P\cdot\left[(1+i)^m-1\right] \Rightarrow \frac{FV\cdot i}{(1+i)^m-1} = P$$

This can be rewritten as: $P = FV\,\frac{i}{(1+i)^m-1}$

Note: This is sometimes called a "**Sinking Fund**."

Example (6.6): A fund of $25,000 needs to be available to a company in 8 years. What monthly payments need to be made to have this money available at that time if the money is compounded monthly at a rate of 6.5%?

Solution: $P = \$25000 \cdot \dfrac{\frac{0.065}{12}}{\left(1+\frac{0.065}{12}\right)^{12(8)} - 1} = \$199.24$

Prior to working through the problem set, write down the definition for the vocabulary words "annuity" and "sinking fund."

Annuity: ____

____

Sinking Fund: ____

____

The Problem Set

General Instructions: Show formulas used to solve problems. Display the appropriate numbers in the formulas, so partial credit can be assigned if the results aren't quite right. Assume all interest rates are yearly rates, even if not explicitly stated. When working with money, round off the final answer to the nearest penny.

1. What will be the future value of an annuity if $7,000 is deposited semiannually at a yearly rate of 7%, compounded every 6 months for 15 years?

2. You are saving towards a house. Towards this end, you put away $600 per month into an investment that is compounded monthly at 9%. How much will you have after 6 years?

3. You need $54,000 in 10 years for a vacation cabin. How much do you need to put aside monthly into an account that is compounded monthly, at an annual rate of 7.5%?

4. You plan to pay $150 per month into your IRA (Individual Retirement Account) for 29 years. The IRA will be an account that guarantees a 7.8% nominal interest rate, compounded monthly. What will you have in your account at the end of that time?

Note: There are restrictions on how much you can deposit in an IRA (or Roth IRA). As of 2003, most people can deposit at most $3,000 each year into their IRA. There is a clause that allows people over the age of 50 to deposit $3,500 in a year. Also, you cannot make a deposit larger than your gross income for the year.

5. Jane paid $200 per month into an IRA account for 20 years that paid 5.5%, compounded monthly. At the end of that time, she "rolled-over" the money that was in her IRA into another account that earned interest continuously at a rate of 6%. She simply left the money, without adding any additional amount, for another 10 years. How much money did she have at the end of the 30 years?

6. You estimate that you will need to replace your car in 5 more years. The car you want will probably cost $25,000. If you think you will get $3,000 for your present car, how much will you need to set aside in order to be able to pay cash for your car? Assume your savings account will pay 5.5%, compounded monthly.

7. For some IRA's you only make annual contributions. If you invest $3,000 annually in an investment that pays 12%, compounded annually, for 10 years, how much will you have?

8. See if you can make a formula that will calculate the amount you will have after 10 years if you put $500 per month into an account that pays continuous interest of 6%. (This will require you to modify the original annuity formula. The original formula always assumes the payment period is the same as the compounding period. Try one of two approaches: (1) remember that 1-month is 1/12 of a year and go back and see how the original formula was developed. Or (2) look at the problems at the end of the previous section and find the monthly interest rate that is equivalent to the continuous interest rate.)

Section 7
Paying off a Loan (Amortization)

Scenario: An item is purchased at a price L, and it is going to be paid off by installment payments over a period of t years. Of course the people "holding the loan" expect to be paid interest on the unpaid balance. Assume that the loan is paid off in m payments (a payment made each period) and that the annual interest rate is r, compounded n times per year.

Forming the Equation: As with the annuity scenario in the previous section, let's look at the amount of money left after each payment is made. Let PMT be the payment made each period. Remember, the balance of our loan increases due to interest in each compounding period. Let $i = \frac{r}{n}$, the interest rate during each compounding period.

We start with a loan of L

$$L$$

$$\downarrow$$

after the first payment, the unpaid principal

$$(1+i)L - PMT$$

$$\downarrow$$

the unpaid principal after the second payment

$$(1+i)[(1+i)L - PMT] - PMT$$

$$\|$$

$$L(1+i)^2 - PMT(1+i) - PMT$$

$$\downarrow$$

multiply by $(1+i)$ and subtract PMT to find the unpaid principle after the third payment

$$L(1+i)^3 - PMT\ (1+i)^2 - PMT(1+i) - PMT$$

$$\downarrow$$

And so on . . .

$$\vdots$$

Note the pattern: There were 3 payments and the largest exponent in the expression is also 3, so . . .

At the end of m payments, when the loan is paid off completely, we would have

$$L(1+i)^m - PMT(1+i)^{m-1} - . . . - PMT(1+i) - PMT$$

and this should equal \$0. Ha, another equation to solve!

$$L(1+i)^m - PMT(1+i)^{m-1} - . . . - PMT(1+i) - PMT = 0$$

$$L(1+i)^m - [PMT(1+i)^{m-1} + . . . + PMT(1+i) + PMT] = 0$$

Continuing on from $L(1+i)^m - [PMT(1+i)^{m-1} + . . . + PMT(1+i) + PMT] = 0$, we add the expression in the "brackets" to both sides of the equation.

$$L(1+i)^m = [PMT(1+i)^{m-1} + . . . + PMT(1+i) + PMT]$$

Now, notice the right side of the equation is the sum of a geometric sequence. By our formula for summing geometric sequences, we get:

$$L(1+i)^m = \frac{PMT(1+i)^m - PMT}{(1+i)-1} = PMT\,\frac{(1+i)^m - 1}{i}.$$

Now, let's solve for *PMT* by dividing and get a formula for *PMT*:

The Payment: If a loan $\$L$ is to be repaid over *m* periods at a periodic interest rate of *i*, then the payment required each period is: $PMT = L\cdot\frac{i(1+i)^m}{(1+i)^m - 1} = L\cdot\frac{i}{1-(1+i)^{-m}}$

The last form of this formula is what is usually given in texts. You get this from the expression in front of it by dividing top and bottom of the fraction by $(1+i)^m$ and remembering that $1 \div b^x$ is b^{-x}.

Example (7.1): Suppose that the Jones' have a mortgage of $124,000 on their new house. The loan is at 7.8% and the payments are made monthly over 30 years. What are their payments? Assume interest is also computed monthly.

Solution: $PMT = \$124000\cdot\frac{\frac{0.078}{12}}{1-\left(1+\frac{0.078}{12}\right)^{-12(30)}} = \$892.64$

(7.2) Interest Rates.

Not only do interest rates change daily, but each day you have many different options. In this example, I assume that we buy a house for $200,000 with a $40,000 down payment. That is, we take out a loan for $160,000.

Each of these five choices is a 30-year fixed rate mortgage. The difference is in the interest rate. The rates vary a full 0.5% based on the number of discount points you pay.

A discount point is an additional closing cost (or credit) equal to 1% of the loan amount. So, in this case, 1 point = $1,600.

Rates from Boeing Employees Credit Union (BECU) on August 29, 2003.

Interest Rate	Discount Points	Origination Fee	Total Closing Costs	Monthly Mortgage Payment	APR
6.375%	-1.625	1.000%	$2,636	$998	6.365%
6.250%	-1.125	1.000%	$3,428	$985	6.287%
6.125%	-0.625	1.000%	$4,220	$972	6.209%
6.000%	0.000	1.000%	$5,212	$959	6.142%
5.875%	0.500	1.000%	$6,003	$946	6.063%

Example (7.3): You've borrowed $4,000 from a friend. He agrees to charge interest of 5%, compounded quarterly, and you're to make 4 payments per year for 5 years. How much is your quarterly payment? How much did you pay in interest for your loan?

Solution: $PMT = \$4000 \cdot \dfrac{\frac{0.05}{4}}{1-\left(1+\frac{0.05}{4}\right)^{-4(5)}} = \$227.28$.

You made 20 payments of $227.28 which will total $4545.63. You will notice this is <u>more</u> than the amount you borrowed. This extra, of course, is the interest you paid your friend! You paid $4545.63 - $4000 = $545.63 in interest.

(7.4) Closing Costs.

When you get a loan to make a purchase, you are buying something that you cannot afford (or are not willing to pay for). The start-up fees you pay for the privilege of making purchases that you cannot afford are called <u>closing costs</u>. On a home mortgage, closing costs must be paid prior to you taking possession of the house. This means that you need money for a down payment AND money for closing costs in order to buy a house.

In the note on Interest Rates **(7.2)**, you probably noticed that the closing costs varied from about $2,600 to over $6,000 depending on the number of points you paid. We will focus in on the fourth case where you pay no points and the closing costs were $5212.

$$\frac{\$5212}{\$160000} = 0.032575 \approx 3.25\%$$

When planning to purchase a home, you can estimate that you will have to pay closing costs equal to about 3% the value of the loan.

Example (7.5): You wish to keep your monthly payments for a new car over a 4-year period down to $250 per month. The interest rate will be 8.8%. How expensive of a car can you afford?

Solution: This means we will have to solve our formula for *L*. Multiply both sides of your formula by the denominator and you get $PMT(1-(1+i)^{-m}) = L(i)$. Now divide by *i*. Letting $PMT = 250$, $i = \dfrac{0.088}{12}$, and *n* = 4(12) = 48 months, we get

$$L = \frac{250\left(1-\left(1+\frac{0.088}{12}\right)^{-4(12)}\right)}{\frac{0.088}{12}} = \$10,084.64 .$$

You can afford a car at this. If the car costs more than that, you'll have to come up with a down payment of some sort.

(7.6) Type of Loans

There are many types of home mortgages. The two most common are fixed rate mortgages and adjustable rate mortgages. On the same day, drastically different rates are offered depending on the type of loan selected.[2]

Samples from BECU are given

Loan Type	Interest Rate	Discount Points	Origination Fee	Total Closing Costs	Monthly Mortgage Payment	APR
30 Year Fixed	6.250%	-1.125	1.000%	$3,428	$985	6.287%
5/1 Yr ARM	5.500%	-1.125	0.500%	$2,579	$908	4.511%

Example (7.7): Suppose Bill saved $500, 000 for his retirement. The retirement fund is locked in an investment paying 7% per year, compounded monthly. However, Bill can make monthly payments to himself from that fund. How large of a monthly payment will use up Bill's retirement fund in 30 years?

Solution: On first glance, this seems like a different problem than our first example. But it really isn't - we can pretend that the retirement fund is simply a loan that we wish to pay off in 30 years. That "darn retirement" fund keeps growing due to interest, just like our loan on that house, but we are making payments to ourselves (rather than to a mortgage company) to use up that fund in a certain amount of time. So:

$$PMT = \frac{\$500{,}000\left(\frac{0.07}{12}\right)}{1-\left(1+\frac{0.07}{12}\right)^{-12(30)}} = \$3{,}326.51 \text{ per month.}$$

Over the 30 years, Bill paid himself 360($3,326.51) = $1,197,544.49! You "gotta" love that interest!!!

(7.8) Fixed Rate Mortgages versus Adjustable Rate Mortgages – The Pros and Cons

from the BECU website

30-Year Fixed Rate

Best choice if:

- You plan on staying in the home long-term.
- You think interest rates will increase.
- You don't expect your income to increase significantly over the coming years.
- You need to qualify for the largest loan possible.

Advantages:

- Fixed rate of interest.
- Level principal and interest payments for the full term of the loan.
- No risk that changing market conditions will increase your monthly payments

Disadvantages:

- You end up paying more in interest charges over the life of the loan.
- Benefits are not realized until after the 10th year.

5-Year Adjustable Rate

Best choice if you want:

- The stability of a fixed monthly payment for first five years of loan.
- To keep your payments low.
- To maximize the amount of loan you qualify for.

Advantages:

- Initial fixed interest rate for 5 full years.
- The rate adjusts periodically thereafter.
- Allows for higher loan amount qualification and enhanced buying power.

Disadvantages:

- It's riskier if you don't expect your income to increase over the initial five-year period to cover the change in monthly payment.
- Interest rate can rise above the current fixed rates over time.

[2] Information on ARMs may be found at http://flightline.highline.edu/dwilson under the most recent Math 111 class.

The Problem Set

General Instructions: Show formulas used to solve problems. Display the appropriate numbers in the formulas, so partial credit can be assigned if the results aren't quite right. Again, assume the interest rates given are nominal (yearly) rates. When working with money, round off the final answer to the nearest penny.

1. You went to "Electronics R Us" and purchased a stereo for $1,300. They charge you 18% annual interest, compounded monthly, to carry your contract. If you make monthly payments and the loan is to be paid off in 2 years, how much is your monthly payment? What did that stereo really cost you?

2. What would be your semi-annual payment if you borrow $13,000 at 6.7% interest for a period of 7 years? (Assume the interest is only compounded semi-annually also. The formulas get a lot more complicated than given in the book if the compounding by the lender is done at a different rate than the payment schedule.)

3. "Bob's Sell-A-Dent" has a wide selection of used cars. You can afford to pay at most $300 per month. Bob offers to carry the loan contract at an annual rate of 11% for a period of 2 years. A car that you wish to buy would cost you $7,000. Is this car within your budget? (Show work to validate your answer) What's the minimum down-payment (to the nearest dollar) that would be required to bring the monthly payments within your budget?

4. Suppose you buy a house and will have a mortgage of $125,000 to pay off over 29 years, paid off on a monthly basis, at an annual interest rate of 7.8%. Find your monthly payment. Find the amount of interest you paid. (Did this surprise you?)

5. If you're allowed to make extra payments on your mortgage without penalty, many financial experts tell you to pay just a little more per month than you have to. Suppose you make an extra $25 per month payment each month over your answer in problem (4.) above. Solve the equation for *t*. (This extra payment will mean you will pay off your loan in a lesser amount months. You will need to use logarithms to solve your equation.) With your increased payment and lesser number of years, how much did you pay total for the house? How much did you save over what you would have paid in problem (4.) above?

6. Suppose you invest $400 per month into an account that pays 6% annual interest, compounded monthly, for 30 years. At the end of that time, how much can you take out of the account monthly so that the account is used up in another 30 years? (Assume the account continues to be compounded monthly at 6%.)

7. You are planning to buy a new car in 4 years that will probably have a price tag in the neighborhood of $32,000. You start having $250 per month taken out of your paycheck that goes into a savings account (compounded monthly at 5.5% interest) for 4 years. You then take out that money (to nearest dollar) and use it for a down payment on the car of your dreams. If you talk the salesman down to a final price of $30,500, **what will be your monthly payment** if the financing is for 5 years, compounded monthly at 8.9%? (Assume you have the payment taken out automatically from your paycheck). Also, **what did you really pay for that car** in terms of actual money out of your paycheck over all those years?

8. Suppose you have $400,000 in a retirement fund that is being compounded continuously at 6%. How much will you take out each month if you plan to use up all the money in 20 years? Hint: This is not the "usual" case, since the compounding is being done continuously, whereas the payments are taken out monthly. Go back to the section on effective interest rate, and use the ideas there to find what <u>monthly</u> interest rate is the same as the continuous interest rate.

Section 8
Pre-Qualifying for a Home Mortgage

While we have seen many examples of annuities and amortization, we have yet to apply this information to you (the reader) directly. So, let us work to answer a very practical question.

Given my present financial situation, what is the largest loan for which I would be eligible?

The answer to this question will vary from person to person based on their income, present assets, current debt, and credit rating. So, the following worksheet is for the individual and is not meant to be discussed or shared with the class.

One last disclaimer prior to beginning the worksheet – I (the author) am not claiming to be an expert in this area. The numbers and formulas I am presenting are accurate to the best of my knowledge, however they should not be acted upon without first consulting a professional in the field.

Pre Qualify Yourself

Qualification issues all revolve around money. In real estate, money normally means mortgage loans. When a lender considers your mortgage application, it has dozens of elements and issues that it must consider prior to approving the loan. These approval guidelines and restrictions are set by the government, related institutions and the lender itself.[3]

The bulk of these restrictions and requirements fall within four categories:

1. Property. The type of property determines down payment requirements and limits the programs available for the borrower. We will assume you are interested in purchasing a house in which you (the owner) intend to live.
2. Assets. The buyer must demonstrate that he or she has enough funds for the down payment, projected closing costs and reserve requirements.
3. Credit Rating. The borrower's credit grade limits the programs available for the buyer. Lower credit ratings often require larger down payments, higher interest rates and more restrictions.[4]
4. Employment and Income. The borrower must document and verify two years continuous employment (although exceptions are allowed). The borrower must also demonstrate that the projected monthly housing payments, plus monthly payments on all other long-term debts, are within the debt-to-income ratio limits given below. (Also see the personal finance note **(0.1)** in the introduction).

[3] The following is an adaptation of http://www.atlastitle.net/literature/04Self_prequalification.htm

[4] In order to qualify for a loan, you must have a good credit rating. Many mortgage brokers currently advertise that they accept applicants with 'less than perfect credit." While this may be true, those applicants will pay a premium for their mortgage. Finally, this is not a section on credit ratings and so credit will not be mentioned from this point onward.

Your Monthly Income and Payment

In loaning you money, lenders (such as banks) are taking on risk and so they want to know whether you can afford monthly mortgage payments prior to giving you a loan. They use two different ratios to decide how much you can afford. The two debt-to-income (DTI) ratios are the Housing (or front-end) ratio and the Total Long-Term Debt (or back-end) ratio. The conventional loan guideline calls for the housing ratio to be no more than 33% of the gross income, and the total long-term debt ratio to be no more than 38% of the gross income. Of course, there are always exceptions and alternatives. Use the following worksheet to find out how much the lenders will lend you.

Example	**Your Debt-to-Income Ratio**
Gross monthly income: G = \$2500 This income must be documented and consistent in order to qualify.	Your gross monthly income: G = _________
Monthly payments toward debt: D = \$150 This includes car payments, monthly installments, student loans, credit card payments, etc.	Monthly payments toward debt: D = ________
Housing percentage: h = 0.32 Where $h = \min\{0.33, 0.38 - \frac{\$150}{\$2500}\}$	Housing percentage: h = _________ Where $h = \min\{0.33, 0.38 - \frac{D}{G}\}$
Maximum housing allowance: H = \$800 Where $H = 0.32 \cdot \$2400$	Your maximum housing allowance: H = _____ Where $H = h \cdot G$
The mortgage payment: PMT = \$640/mo Where $PMT = 0.8 \cdot \$800$	Your mortgage payment: PMT = __________. Where $PMT = 0.8 \cdot H$

The Actual Loan Amount

So far, we have determined the maximum amount PMT a lender will allow you to put towards a mortgage payment each month. However, this is different, although related, than the size of the loan. To find the loan you qualify for, work through the following steps.

Example	**Pre Qualify Yourself**
The current interest rate: r = 0.0625 (6.25%)	The current interest rate: r = __________. check at http://flightline.highline.edu/dwilson
Number of periods per year: n = 12	Number of periods per year: n = 12
Interest per period: i = 0.005208 Where $i = 0.0625/12$	Interest per period: i = __________ Where $i = r/n$
Years to pay back the loan: t = 30yrs You decide the number of years	Years to pay back the loan: t = ______ Traditionally, 15 or 30 years.

Total payments: m = 360 Where $m = 12 \cdot 30$	Your total payments: m = ______ Where $m = 12 \cdot t$
The loan: L = \$103,944 Where $L = \dfrac{\$640\left(1-(1+0.005208)^{-360}\right)}{0.005208}$	Your loan: L = __________ Where $L = \dfrac{PMT\left(1-(1+i)^{-m}\right)}{i}$

So, you now know the maximum loan for which you qualify with the given parameters . . . well, almost. It takes more than an income to get a loan; you also must have cash on hand for a down payment (or pay a higher interest rate). So, what kind of down payment will you need to make? What follows is an overview of down payments and assets. Once the explanation is complete, we will continue our example/worksheet.

Property Down Payment Requirements

Residential mortgage lenders have differing restrictions, depending on the property type. One of these restrictions is the down payment requirement.

The most important consideration regarding property and down payment requirement is the property's appraised market value. The loan is limited to a percentage of the value or price of the property; whichever is lower. Whatever the loan cannot cover is basically the down payment. A 95% *LTV* loan, for example, requires a 5% down payment.

If the property is worth more than the selling price, the buyer is getting a bargain. However, if the property appraises for less than the sale price, then the buyer must make up the difference or renegotiate.

Traditionally, the down payment on a single-family home was 20% of the value of the property. So, on a \$200,000 house, you could expect to make a down payment of \$40,000. However, most first time buyers do not have assets large enough to afford this kind of down payment. To accommodate these buyers, lenders allow conventional loans up to 97% of the purchase price (3% down) although they charge a higher interest. However, the recommended minimum down payment for single-family homes is 3%-10%.[5]

So, what assets do you have to use for a down payment?

Assets

First of all, you do not need to list all of your assets. Hard assets such as cars, jewelry and personal effects are rarely accounted for in the pre-qualification process. The main asset requirements regard your liquid assets or cash deposits. All deposits to be used for a mortgage transac-

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[5] There are other types of loans that do not require a down-payment or closing costs. An example of this is the 103% loan that covers the entire purchase price and the closing costs. The reason these were not mentioned was that they may require paying a significantly higher interest rate. Furthermore, this means you owe more on the loan than you can (may) potentially sell the asset for (ignoring appreciation).

tion should be documented in a bank account with monthly statements for at least three months. Acceptable liquid assets include the following:

- Savings and checking accounts
- Stocks, bonds, mutual funds and other securities
- IRA accounts and loans
- Profits from sale of property
- Gifts

Lenders (such as banks) are wary of any sudden and recent large deposits, as they often indicate unacceptable cash. The reason for these asset documentation requirements is that borrowed funds can decrease the borrower's net worth, increase their debt-to-income ratios and jeopardize their financial stability with future payment demands.

The borrower must show that there are sufficient liquid assets to qualify for the loan program requested. Use the following worksheet to estimate the largest house for which you qualify.

Note: In the following example, I assume that closing costs and "pre-paids" are 3.25% of the loan. This may be a slight overestimate. However, you may expect to pay at least 3% in closing costs.

Example	**Pre Qualify Yourself**
Recall the L = \$103,944	Recall that L = __________
Amount of cash available to use to purchase a home: C = \$7,000.	Amount of cash available to use to purchase a home: C = ____________.
Closing costs and pre-paids: f = \$3378 Where $f = 0.0325 \cdot \$103,944$ See note **(7.4)**.	Your closing costs and pre-paids: f = ________ Where $f = 0.0325 \cdot C$
The down payment: d = \$3622 Where $d = \$7000 - \$3378$	Your down payment: d = __________. Where $d = C - f$
The maximum buying power: P = \$107,566 Where $P = \$103,944 + \$3,622$	The maximum buying power: P = __________ Where $P = L + d$
The ratio of loan to the value: LTV = 0.97 Where $LTV = \$103,944 / \$107,566$ This number ranges from the risky 0.97 to a conservative 0.80. If you were paying cash for the house, LTV would be zero since there would not be a loan.	Your ratio of loan to value: LTV = _________. Where $LTV = L/P$ If your loan to value is too high, then your only options are to save more or borrow less.

The big question: *Is the P you calculated large enough to buy the house you want?*

If the answer to the above question is YES, then you may (just maybe) go ahead with the purchase of your home. However, if the answer is "*NO*", then you are left with a couple of choices.

- You can save more money to use toward the down payment and closing costs.
- You could accept a higher interest rate in exchange for discount points. That is, you would pay less closing costs, which gives you more for a down payment. However, the interest rate would be higher so you will pay more over the life of the loan.
- You can choose a different kind of loan. In the example, I choose a 30-year fixed-rate conventional mortgage at 6.25%. However, if I was willing to accept the higher risk of an adjustable rate mortgage, I could get a 5-year ARM at 5.5%.
- You can wait for interest rates r to decline. However, be forewarned that interest rates can increase just as easily as decline.

So, play with the numbers so that you can find the amount of cash you need and the interest rate you must find to give you the house of your dreams.

Appendices

Appendix 1: Summary of Formulas

Be sure to ask your instructor which of these must be memorized!!!

(9.1) $S_n = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \ldots a_1 r^{n-1} = \frac{a_1 r^n - a_1}{r-1} = a_1 \frac{r^n - 1}{r-1}$

(9.2) $A = P(1+i)^m$ where $i = \frac{r}{n}$

(9.3) $A = Pe^{rt}$

(9.4) $r_{eff} = (1+i)^n - 1$ and (9.5) $r_{eff} = e^r - 1$

(9.6) $FV = P\frac{(1+i)^m - 1}{i}$ and (9.7) $P = FV\frac{i}{(1+i)^m - 1}$

(9.8) $PMT = L\frac{i}{1-(1+i)^{-m}}$

Note: Some texts make distinctions between **ordinary annuities** and **annuities due.** The formulas given in this handout is equivalent to the **ordinary annuity** concept. (The difference lies in whether payments are taken before the interest is calculated or after - payments at the first of each month or at the end - it's a slight difference of counting the value of *n*.) If the problems in a future class (or if you are doing extra problems in some other text) indicates it's an annuity due, then just use the appropriate formula given to you for that concept.

Appendix 2: Miscellaneous Problems

These are in no particular order, much as you might expect in a test. Use the appropriate formula(s) for each of the problems and answer the question. These are not comprehensive as to all the possible variations of questions, but should give you a basic problem of each topic discussed.

1. A deposit of $100 is made at the end of each month in an account that pays 10% interest, compounded monthly. Find the balance in the account after fifteen years.

2. You want to borrow money to buy furniture for your living room. You obtain a loan for $5000 to be paid back monthly at 8.5% interest over the next 8 years. Find the monthly payment and the total amount of interest you will pay over the life of the loan.

3. Find the effective interest rate for a savings account that pays 4.5% compounded quarterly.

4. Compare the interest rates for savings accounts at the following banks:

(a) Bank A offers 3.8% compounded daily
(b) Bank B offers 3.82% compounded weekly
(c) Bank C offers 3.83% compounded monthly
(d) Bank D offers 3.85% compounded semi-annually.
(e) Which bank would you use? (which is the "best deal"?)

5. You are working at a job where you have an IRA account in which you deposit $1000 each quarter at 8% for 15 years. (Interest is compounded quarterly also.) You then leave the company and rollover the money into a savings account that pays 6% continuously for the next 10 years (no additional money added). It is now time to retire, how much do you have saved?

6. A total of $12,000 is invested at an annual interest rate of 9%. Find the balance after 5 years if it is compounded

(a) quarterly.
(b) continuously.

7. You are planning on buying a vacation condo and the price tag is $70,000 (including taxes, etc.). If you finance it on a 20-year loan at 6.9%, compounded monthly, how much will your monthly payment be?

8. You want to set aside, in a bank account, an amount of money each month so that you can buy a new car for $20,000 in 3 ½ years. Assuming the annual interest rate is 5.7%, compounded monthly, how much are your monthly payments into your account?

Appendix 3: Selected Solutions/Hints

(Some solutions are only partially shown)

Section 1: Sequences

1. Parts a, c, and d

(a) $a_1 = 3 + 2(1)^2 = 5$, $a_2 = 3 + 2(2)^2 = 11$, $a_3 = 3 + 2(3)^2 = 21$, $a_4 = 3 + 2(4)^2 = 35$, and so on where the pattern is: 5, 11, 21, 35, 53, 75

(c) 1, 4, 27, 256, 3125, 46656

(d) $f_1 = (-1)^{1-1}\frac{1+1}{1^2} = 2$ and $f_2 = (-1)^{2-1}\frac{2+1}{2^2} = -\frac{3}{4}$, and so on where the pattern is:

2, -3/4 , 4/9 , -5/16 , 6/25, -7/49

(the pattern is easier to see if the answers are left as fractions!)

2b. (i) An easy pattern: The next 3 terms are 5/6, 6/7, 7/8.

(ii) $f_n = \frac{n}{n+1}$ (*n* gives you 1,2,3,4,5,6,7 for numerator, and denominator is 1 larger)

3. Start with 5, and then multiply by 3 to get successive terms:

5, 15, 45, 135, 405, 1215

The formula for this sequence is $a_n = 5(3)^{n\text{-}1}$. Hence $a_{17} = 5(3)^{16} = 215233605$.

$$S_{12} = \frac{5(3)^{12} - 5}{3 - 1} = 1328600$$

7. - 2, 4, -8, 16, -32, 64

$$S_8 = \frac{-2(-2)^8 - (-2)}{-2-1} = \frac{-510}{-3} = 170$$

$$S_{29} = \frac{-2(-2)^{29} - (-2)}{-2-1} = -357{,}913{,}942$$

9. If the loan is L, the remaining new loan balance is $L - .12L = 0.88L$.
So just multiply by .88 to get the next month's balance. Hence we have a geometric sequence whose ratio is .88. The balance after n payments is $B_n = 4400(.88)^{n-1}$. So, after 1 year (12 payments) the balance is $4400(.88)^{11}$ = \$1,078.36 (By the way, you'd be correct if you said that $B_n = 5000(.88)^n$). If we want to know when we have \$10 left, solve

$$10 = 4400(.88)^{n-1} \Rightarrow 10/4400 = .88^{n-1} \Rightarrow \log(1/440) = (n-1)\log .88$$

$$(n-1)\log .88 \Rightarrow \log(1/440) / \log .88 = n-1 \Rightarrow n - 1 \approx 47.6 \Rightarrow n \approx 48.6 \text{ months}$$

If you similarly solve the equation for \$1, you will get $n \approx 66.6$ months. (From a practical point of view, at some point you just say, "I'll pay off the balance" rather than dragging this out forever.)

Section 2: Arithmetic Series

Section 3: Geometric Series

Section 4: Compound Interest

1. $A = \$5000\left(1 + \dfrac{0.06}{12}\right)^{12(15)} = \$12,270.47$

3. (c) $A = \$10000\left(1+\dfrac{0.08}{12}\right)^{12(20)} = \$\ 49,268.03$

(f) $A = \$10000e^{0.08(20)} = \$49,530.32$

5. $\$600000 = P\left(1 + \dfrac{0.045}{12}\right)^{12(15)} \Rightarrow P = \dfrac{\$600000}{\left(1 + \dfrac{0.045}{12}\right)^{12(15)}} = \$305,879.77$

7. A = \$17,425.64

9. ≈7 yrs 10 months

11. ≈ 128.31 months ≈ 10.7 years

Section 5: Effective Annual Interest

1. $r_{eff} = \left(1+\dfrac{0.056}{365}\right)^{365} - 1 \approx 0.0576 = 5.76\%$

3. Solve $0.072 = e^r - 1$. (show work) $\Rightarrow r \approx 6.95\%$

7. Solve $\$1\left(1+\frac{r}{12}\right)^{12}=\$1e^{0.07}$. (The effect on \$1 for one year should be the same).

Taking the 12th root, $1+\frac{r}{12}=\left(e^{0.07}\right)^{1/12}$ $\Rightarrow$ you finish the work.

Section 6: Annuities

1. $FV=7000\frac{(1+0.07/2)^{2(15)}-1}{0.07/2}=\$361{,}358.74$

3. Solve $54000=P\frac{(1+0.075/12)^{12(10)}-1}{0.075/12}$. You should get P = \$303.49 .

5. \$158,752.97

7. \$52,646.21

Section 7: Amortization (Loans)

1. $Pmt=1300\frac{0.18/12}{1-(1+0.18/12)^{-12(2)}}=\$64.90$

Since you made 24 payments of \$64.90, the stereo cost 24(64.90) or \$1,557.60. The extra over the \$1,300 is the interest you paid.

3. We need an approximate down payment of \$564.

5. We need to solve the equation below for *m*. This is another job for logarithms.

$$\$932.73=\$125{,}000\left(\frac{\frac{0.078}{12}}{1-\left(1+\frac{0.078}{12}\right)^{-m}}\right)$$

$$\Rightarrow\ \$932.73\left(1-\left(1+\frac{0.078}{12}\right)^{-m}\right)=\$125{,}000\left(\frac{0.078}{12}\right)$$

$m\approx 316.2$ months ($\approx$26 years 4 months)

(take the log of both sides and solve for *m*.)

316 months at \$932.73 per month gives a total of \$294,742.68, a savings of \$21,147.36. If you use 316.2 for your value of m, you'll get an answer of \$20,960.81 for savings.

7. You will save \$13,388.21. You take out \$13, 388.00 for your down payment.

What you still owe is \$30,500-\$13,388 = \$17,112.

This will be the loan you take out. Your monthly payment is $354.39.

Appendix 2: Miscellaneous Problems

1. $41,447.03

3. $r_{eff} \approx 0.04576...4.58\%$

5. $FV = 1000\dfrac{(1+0.08/4)^{15(4)}-1}{0.08/4} = \$114,051.54$ This is put in a savings account

(assuming no additional monthly payments)

$$A = \$114,051.54e^{0.06(10)} = \$207,815.46$$

7. Payment is $538.52